UKMT

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Julia Robinson

Sheet 1

Solutions and comments

This programme of the Mentoring Scheme is named after Julia Robinson (1919–1985).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Robinson_Julia.html for more information.

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1. This is the first of the Robinson series of mentoring sheets, named after *Julia Robinson*, a mathematician who developed the important mathematical fields of *complexity theory* and *computability theory*, proving some of the important results. What can you find out about her, and her areas of work?

Solution

There are lots of interesting things to find out about Julia Robinson: here are a few things I found out.

Julia Robinson was born in 1919 in Missouri in the USA, but moved to California when she was young. She went to San Diego State University and then to Berkeley.

Her main areas of mathematical interest were in computability theory, which is at the boundary of mathematics and computer science, determining whether special types of algorithm can exist.

The key mathematical result proved by Julia Robinson was that any Diophantine equation (a polynomial equation with a number of unknowns where you are looking for integer solutions) can always be transformed into an equation with at most nine unknowns, such that either both equations have solutions or neither of them do. This was a major step in the study of *Hilbert's tenth problem*, which asks for an algorithm that determines whether a Diophantine equation has integer solutions.

Julia Robinson became the first female mathematician elected to the US National Academy of Sciences in 1976 and the first woman president of the American Mathematical society in 1983. She died in 1985.

2. How many four digit integers, written in base 10, only use the digits 0, 1, 4 and 9. The digits can be repeated, and do not all have to be used.

Answer 192

Solution

Let's start by thinking about the first of the four digits. We know that the first digit cannot be 0, since then the number would have less than four digits. This leaves three possibilities: 1, 4 and 9

Then, for each of the other digits, there are four options, since we can have any of the digits 0, 1, 4 or 9. This means that there are four options for each of the other positions.

Therefore, there are $3 \times 4 \times 4 \times 4 = 192$ possible options for the four digits, as we are allowed to repeat the digits in the number.

3. A car made a journey of 144 miles, stopping for one hour along the way. Had it travelled at an average speed 4 mph faster and stopped for one and a half hours, it would have taken the same time.

What was its average speed?

The averages in this question should be taken only over the time at which the car is moving, not while it was stopped.

Solution

Let the average speed of the car be v, measured in miles per hour. We know that average speed $= \frac{\text{total distance}}{\text{time}}$, so we can write the time as time $= \frac{\text{total distance}}{\text{average speed}}$.

The time, in hours, for the journey is $1 + \frac{144}{v}$. The alternative journey would have taken $1.5 + \frac{144}{v+4}$ hours. Since these times are the same, we equate the two expressions, so:

$$1 + \frac{144}{v} = 1.5 + \frac{144}{v+4}$$

We subtract 1 from each side, to obtain

$$\frac{144}{v} = \frac{1}{2} + \frac{144}{v+4}$$

We clear the denominators by multiplying both sides by $2v(v+4)$, giving

$$288(v+4) = v(v+4) + 288v$$

We can then subtract $288v$ from both sides, which gives

$$v(v+4) = 1152$$

Or

$$v^2 + 4v - 1152 = 0$$

> We have reached a quadratic equation for v, which we would like to be able to factorise, so we'd like to be able to guess one of the factors. We know that we are looking for two numbers that differ by four and multiply to 1152.
>
> We'd expect these numbers to be quite close to the square root of 1152, which is $\sqrt{1152} \approx 33.94$. By trying some integers near this, we can find that $v = 32$ is a potential solution.

We can factorise this equation as

$$(v - 32)(v + 36)$$

This means the only possible values of v are 32 and -36. But we know that the value of v must be positive, so the only possible solution is 32.

We should still check that this is a valid solution. The original journey would have taken $\frac{144}{32} + 1 = 5.5$ hours. The alternative would have taken $\frac{144}{32+6} + 1.5 = 5.5$ hours also, so 32mph is the correct speed.

4. (a) Suppose that there are 1000 people in a room. Explain why two of them must share a birthday. What is the smallest number of people in the room to guarantee that two of them share a birthday?

(b) Suppose that S is a set of 10 integers. Prove that there are two integers in S which have a difference that is divisible by 9.

Answer

(a) 367

Solution

(a) There are only 366 possible birthdays that the people in the room can have. Therefore, as long as there are more people in the room than this, two of them must have the same birthday. If there were 366 people in the room, there could be one with each birthday, so the smallest possible number of people is 367.

> This result is known as the *pigeonhole principle*, which says that if you have $k+1$ pigeons to place into k pigeonholes, then at least one pigeonhole has more than one pigeon in it. You can replace pigeon and pigeonhole with any other objects - in our case it was people and birthdays.

(b) There are nine possible remainders that an integer can have when divided by 9, and we have ten integers. This means, by the pigeonhole principle, that there are at least two of our numbers with the same remainder. (Here the remainders are the pigeonholes and the integers are the pigeons.)

Then, consider the two numbers with the same remainder on division by 9. Their difference must then be divisible by 9, which is exactly what we wanted.

5. What is the area of a regular dodecagon (polygon with twelve sides) that has diameter 2?

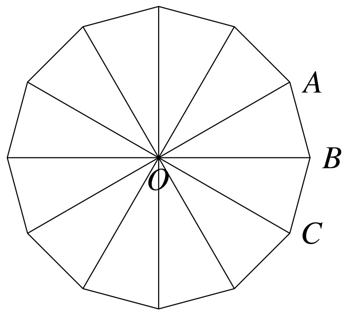
The diameter is the diameter of the circle that passes though all of the vertices of the dodecagon.

Can you use this result to find a lower bound for the value of π?

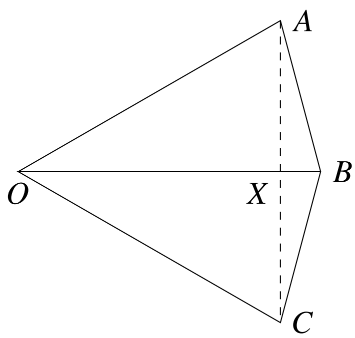
Answer $3, \pi \geq 3$

Solution

We can divide the dodecagon up into 12 triangles as follows:



Then, we want to calculate the area of each of these triangles, such as OAB and OBC.



Now, since the dodecagon has 12 sides, each of the angles at O must be $\frac{360°}{12} = 30°$. This means $\angle AOB = \angle BOC = 30°$. To find the area of triangle AOB, we know that $OB = 1$, so we need to find the height AX.

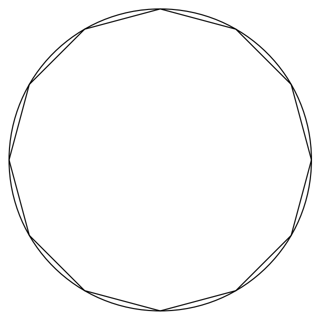
> There are a number of different ways to do this, including calculating $\sin 30°$, which you might know, or you can calculate using half an equilateral triangle. The method below is similar, but fits nicely into the surrounds of the question.

Condiser the triangle OAC. This has angle $\angle AOC = 60°$. However, it is also an isosceles triangle, as $OA = OB = 1$. This means it must be an equilateral triangle. This means $AC = 1$ also.

Now, if X is the point where AC intersects OB. Since $\angle XOA = 30°$ and $\angle OAX = 60°$, we can use angles in triangle OAX to deduce that $\angle AXO = 180° - \angle XOA - \angle OAX = 180° - 30° - 60° = 90°$. This means that X is the midpoint of AC also, since OAC is equilateral, so $AX = \frac{1}{2}$.

Then, we can calculate the area of triangle OAB as $\frac{1}{2} \times \text{base} \times \text{height}$. This is $\frac{1}{2} \times OB \times AX = \frac{1}{2} \times 1 \times 12 = \frac{1}{4}$. As there were 12 of these triangles, the total area of the dodecagon is $12 \times \frac{1}{4} = 3$

To find a lower bound for the value of π, consider drawing a circle around the dodecagon, which has radius 1.

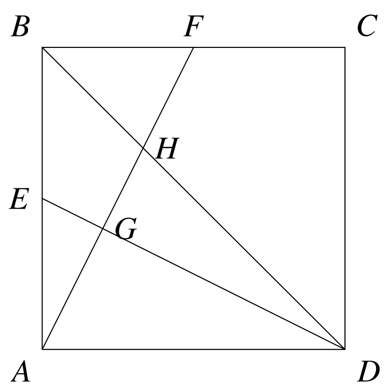


The circle has radius 1, so area $\pi r^2 = \pi$. The dodecagon fits within the circle, so must have a smaller area, which tells us that $3 \leq \pi$, giving the lower bound that we wanted.

6. $ABCD$ is a square of side length 2 units. E is the midpoint of AB and F is the midpoint of BC. AF and DE meet at G. AF and BD meet at H. Find the area of the quadrilateral $BEGH$.

Answer $\frac{7}{15}$

Solution



> There are lots of different approaches that you can take to solve this question. The strategy that I went for was to calculate the area of triangle BHA, and then subtract away from this the area of triangle EGA. Can you find some others?
>
> To do this, you need to say something about the locations of G and H. To do this I am going to use coordinates, but again there are other possible methods.

Let us use a coordinate system, where the origin is at A, and the x-axis along the edge AD. Then, the

coordinates are as follows:

Point	Coordinate
A	$(0,0)$
B	$(0,2)$
C	$(2,2)$
D	$(2,0)$
E	$(0,1)$
F	$(1,2)$

The line AF passes through $(0,0)$ and $(1,2)$, so has gradient $\frac{2-0}{1-0} = 2$. This means it has equation $y = 2x$. The line ED passes through $(0,1)$ and $(2,0)$, so has gradient $\frac{0-1}{2-0} = -\frac{1}{2}$. This means it has equation $y = 1 - \frac{1}{2}x$. G lies at the intersection of these lines, so if it has coordinates (x, y), these satisfy both the equations.

Combining these gives $2x = 1 - \frac{1}{2}x$. Collecting like terms and multiplying out gives $x = \frac{2}{5}$. Then $y = 2x = 2 \times \frac{2}{5} = \frac{4}{5}$, so G has coordinates $\left(\frac{2}{5}, \frac{4}{5}\right)$.

The line BD passes through $(0,2)$ and $(2,0)$, so has gradient $\frac{0-2}{2-0} = -1$. This means that it has equation $y = 2 - x$.

The point H lies on both the lines BD and AF, and thus, if it has coordinates (x, y), then these satisfy both $y = 2x$ and $y = 2 - x$. Combining these gives $2x = 2 - x$, which rearranges to give $x = \frac{2}{3}$. Then, $y = 2x = 2 \times \frac{2}{3} = \frac{4}{3}$, so H has coordinates $\left(\frac{2}{3}, \frac{4}{3}\right)$.

Now, consider triangle BHA. This has base $AB = 2$ and height equal to the x-coordinate of H, which is $\frac{2}{3}$. Therefore the area of BHA is $\frac{1}{2} \times 2 \times \frac{2}{3} = \frac{2}{3}$.

Considering triangle EGA. This has base $AE = 1$, and height equal to the x-coordinate of G, which is $\frac{2}{5}$. Therefore the areas is $\frac{1}{2} \times 1 \times \frac{2}{5} = \frac{1}{5}$.

Then, the area of $BEGH$ is the area of BHA subtract the area of EGA. This is $\frac{2}{3} - \frac{1}{5} = \frac{7}{15}$.

7. Four teams, A, B, C and D, play each other in a local mini-league. A team scores 5 points for a win, 2 points for a draw, and 0 points for a loss, together with an extra 1 point for each goal scored by that team. Last season, every team scored at least one goal in each match. Each team played each other team exactly once. The final results for the season are shown in the table.

Team	Points
A	17
B	13
C	8
D	6

Find the score in each match played. Is there more than one possible set of scores?

Answer The only possible results are:

Results		
A	1-1	B
A	2-1	C
A	2-1	D
B	1-1	C
B	2-1	D
C	2-2	D

Solution

We know that each team scored at least one goal in each match. Therefore, we subtract three points from each team (one for each game) without affecting the results. The amended table, which we will use for the remainder of the solution, is:

Team	Points
A	14
B	10
C	5
D	3

A win is worth five points, but if a team were to win, then it always needs to score as well. The winning team would be awarded at least six points in total for the match. As C and D have less than six points each, they must have drawn, which is worth two points to each player.

Then D has only one point remaining as two were accounted for above, so cannot have drawn any other games, and therefore both A and B must have beaten D. This means that we know the following results:

Results		
A	beat	D
B	beat	D
C	drew with	D

Subtracting off the minimum points for each result (6 for a win and 2 for a draw) leaves the following points not accounted for:

Team	Points
A	8
B	4
C	3
D	1

Neither B nor C have sufficient points to have won against the other, so they must have drawn. This leaves just one point for C, which is insufficient for a draw with A, so C must have lost to A. Then the results

must have been:

Results		
A	beat	C
A	beat	D
B	drew with	C
B	beat	D
C	drew with	D

Subtracting the minimum points gives:

Team	Points
A	2
B	2
C	1
D	1

Then, the only possible result between A and B is a 0-0 draw, as they scored only two more points each. There are no extra points, so A must have won both games 1-0 and B must have had a 0-0 draw with C and a 1-0 win over D.

Then the results are:

Results		
A	0-0	B
A	1-0	C
A	1-0	D
B	0-0	C
B	1-0	D
C	drew with	D

The remaining points are:

Team	Points
A	0
B	0
C	1
D	1

This means that C and D must have both scored an extra point for a goal in their draw against each other, so this game finished 1-1.

Finally, we have to remember to add the one goal to each game that we took away at the start of this

solution, which gives final results of:

Results		
A	1-1	B
A	2-1	C
A	2-1	D
B	1-1	C
B	2-1	D
C	2-2	D

This is the only possible outcome, as we have deduced each result as the only possible one at each stage.

8. Suppose that p is an odd prime. How many different possible remainders are there when dividing a square number by p?

Answer $\frac{p+1}{2}$

Solution

We are considering the remainder when dividing by p, so we can work modulo p throughout this question.

> Ask your mentor if you haven't met modular arithmetic before. If you were on the Cartwright scheme last year then it was introduced on question 1 on sheet 3 - go back and have a look if you can't remember.

Suppose that we have two squares x^2 and y^2 with the same remainder when divided by p. This tells us that $x^2 \equiv y^2 \pmod{p}$, which we can rewrite as

$$x^2 - y^2 \equiv 0 \pmod{p}$$

Now, we can factorise this as $(x+y)(x-y) \equiv 0 \pmod{p}$, which means that p divides $(x+y)(x-y)$.

Now, as p is a prime, p must divide one of the factors. This means that either $x - y \equiv 0 \pmod{p}$ or $x + y \equiv 0 \pmod{p}$. This tells us that we get the same remainder if x and y have the same value modulo p, so there are only p values that we need to think about: the remainders modulo p.

The only way these can give the same remainder when squared is when they add to a multiple of p, from the factorisation above. This pairs the remainder x as having the same remainder as $p - x$. These are distinct unless $p - x \equiv x \pmod{p}$, which is the same as $2x \equiv 0 \pmod{p}$. This means p divides $2x$, but as p is odd, this means p divides x, so $x \equiv 0 \pmod{p}$. The other $p - 1$ pair up, so we get $\frac{p-1}{2} + 1 = \frac{p+1}{2}$ different remainders for squares modulo p.

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Sheet 2

Solutions and comments

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1. There are 27 students in a class, each of whom likes at least one of dogs, cats and rabbits.

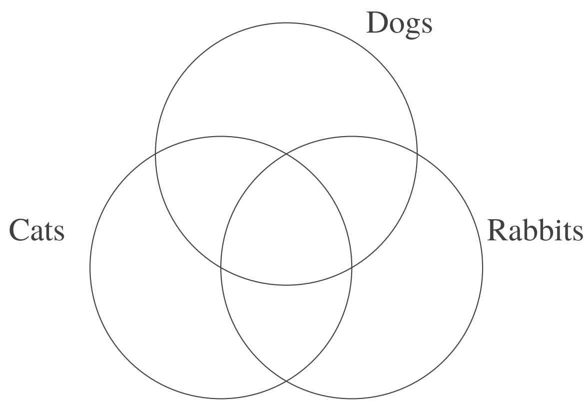
- 16 students like cats.
- 15 students like dogs.
- 9 students like rabbits.
- 7 students like only cats.
- 12 students like at least two animals.

How many students like all of cats, dogs and rabbits?

Answer 1

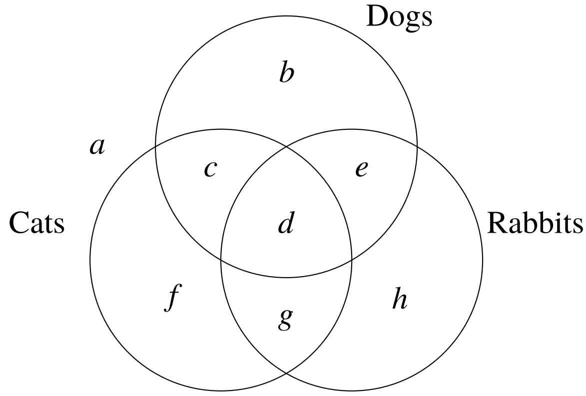
Solution

A good way of visualising this is to think about a Venn diagram with the different parts labelled.



One approach here would be to label all the regions that you have formed and then to use algebra to solve the problem.

Let us label the regions as follows:



We know that all 27 students like at least one animal, and 7 students like only cats. This means that there are 20 students who like at least one of dogs or rabbits. Algebraically, this says that $b + c + d + e + g + h = 20$

Of these 20 students, 15 like dogs, so 5 like either cats only or cats and rabbits only, so $g + h = 5$. Likewise, 9 like rabbits, so $b + c = 11$. This means that the other 4 out of the 20 students must like both rabbits and dogs. Hence $d + e = 4$.

Also, we know that 16 students like cats, of whom 7 like only cats. This means that 9 also like at least one other pet, so $c + d + g = 9$. But we know that 12 students like at least two pets, so $c + d + e + g = 12$. Combining these gives $e = 3$.

But then $d + e = 4$ and $e = 3$ together imply that $d = 1$. This means that one student likes all three pets.

2. (a) Suppose that real numbers x and y satisfy the equation $x^2 + y^2 = 4$. Suppose further that we locate the point (x, y) on the coordinate grid; how far away from the origin is it?

What shape does the curve $x^2 + y^2 = 4$ have? Make sure you describe it precisely.

(b) What shape does the curve $(x - a)^2 + (y - b)^2 = R^2$ form? Describe it precisely.

(c) By changing the value of R, it is possible to change the number of solutions (x, y) to the simultaneous equations:

$$x^2 + (y - 3)^2 = 1;$$
$$(x - 4)^2 + y^2 = R^2.$$

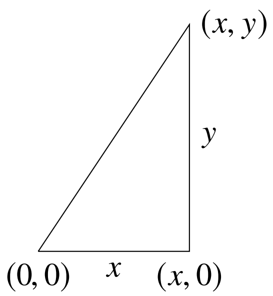
What numbers of solutions is it possible to obtain?

Answer

(a) 2 units. A circle of radius 2 with centre $(0, 0)$.
(b) A circle of radius R with centre (a, b).
(c) 0, 1 or 2.

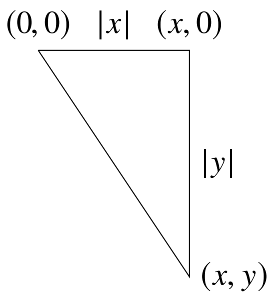
Solution

(a) Let us start by thinking about the case where $x, y \geq 0$. Then we can form the following triangle by considering the coordinate grid.



Since this is a right angled triangle, we can use Pythagoras' Theorem to determine how far (x, y) is from the origin. The distance is $\sqrt{x^2 + y^2} = \sqrt{4} = 2$. This means we know that (x, y) is at distance 2 from the origin.

Now, in the cases where x or y could be either positive or negative, we can do very much the same argument using $|x|$ (which denotes whichever of x and $-x$ is positive) and $|y|$.

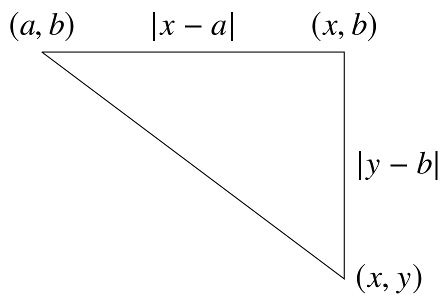


In exactly the same way, the distance between (x, y) and the origin is $\sqrt{|x|^2 + |y|^2}$. But, since $|x|$ is equal to either x or $-x$, $|x|^2 = x^2$. This tells us that (x, y) is at distance 2 from the origin.

This argument tells us that every point satisfying the equation $x^2 + y^2 = 4$ is at distance 2 from the origin. The same logic in reverse tells us that, if a point (x, y) is at distance 2 from the origin, then the expression $\sqrt{x^2 + y^2}$ must equal 2, so $x^2 + y^2 = 4$.

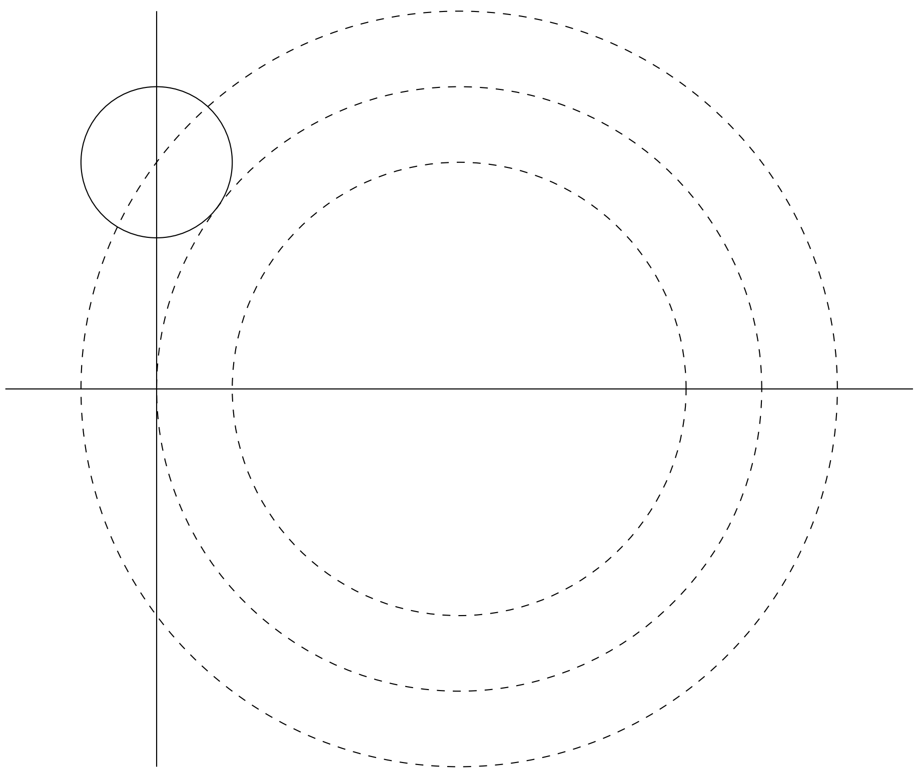
Hence, the points satisfying $x^2 + y^2 = 4$ are exactly those at distance 2 from the origin. Consequently they form a circle with centre $(0, 0)$ and radius 2.

(b) This equation looks similar to the previous one, but involves $x - a$ and $y - b$. This corresponds to translating the configuration in the previous problem by $\begin{pmatrix} a \\ b \end{pmatrix}$ and so we want to consider distances from (a, b).



This labelled diagram tells us that the distance from (a, b) to (x, y) is $\sqrt{(x - a)^2 + (y - b)^2}$. Therefore the points satisfying $(x - a)^2 + (y - b)^2 = R^2$ are precisely those points at a distance R from (a, b). This means that they form a circle of radius R with centre (a, b).

(c) Using the previous parts, we know that these equations correspond to a circle of radius 1, centred at $(0, 3)$ and a circle of radius 2 centred at $(4, 0)$. The number of solutions to the simultaneous equations is then the number of points lying on both circles. The circles can intersect in 0, 1 or 2 places, depending on the value of R.



If you want to, you can calculate the radii corresponding to these different solutions.

- If $R < 4$ or $R > 6$, then there are no solutions.
- If $R = 4$ or $R = 6$, then there is one solution.
- If $4 < R < 6$, then there are two solutions.

3. (a) Suppose that numbers a_1, a_2, b_1 and b_2 satisfy $a_1 \leq a_2$ and $b_1 \leq b_2$. Prove that

$$a_1 b_1 + a_2 b_2 \geq a_1 b_2 + b_2 a_1.$$

In what circumstances is this inequality actually an equality?

(b) Suppose that we have two lists of numbers $a_1 \leq a_2 \leq a_3 \leq a_4$ and $b_1 \leq b_2 \leq b_3 \leq b_4$. Your job is to pair up the a's with the b's so the sum of the product of the pairs is as large as possible. What is the largest value you can obtain?

This result is known as the rearrangement inequality. Can you see how it might generalise?

(c) Suppose you have a collection of numbers c_1, c_2, c_3, c_4. Show that the mean of the squares is greater than the square of the means, that is

$$\frac{c_1^2 + c_2^2 + c_3^2 + c_4^2}{4} \geq \left(\frac{c_1 + c_2 + c_3 + c_4}{4}\right)^2.$$

Answer

(a) When $a_1 = a_2$ or $b_1 = b_2$.

(b) $a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$.

Solution

(a) Consider the expression $a_1 b_1 + a_2 b_2 - a_1 b_2 - a_2 b_1$. We can factorise this as $(a_2 - a_1)(b_2 - b_1)$. But, since we know that $a_2 \geq a_1$ and $b_2 \geq b_1$, both of these factors are non-negative. This means that their product is non-negative, so $a_1 b_1 + a_2 b_2 - a_1 b_2 - a_2 b_1 \geq 0$. Rearranging this gives $a_1 b_1 + a_2 b_2 \geq a_1 b_2 + a_2 b_1$, which is what we wanted to show.

To get equality, we need one of the factors to equal zero, so we must have either $a_1 = a_2$ or $b_1 = b_2$.

Alternative

Since $a_2 \geq a_1$, we can write $a_2 = a_1 + x$, where $x \geq 0$. Likewise, we can write $b_2 = b_1 + y$ with $y \geq 0$. Then,

$$\begin{aligned} a_2 b_2 + a_1 b_1 - a_2 b_1 - a_1 b_2 &= (a_1 + x)(b_1 + y) + a_1 b_1 - (a_1 + x) b_1 - a_1(b_1 + y) \\ &= a_1 b_1 + a_1 y + b_1 x + xy + a_1 b_1 - a_1 b_1 - x b_1 - a_1 b_1 - a_1 y \\ &= xy. \end{aligned}$$

Then, since x and y are both non-negative, their product is also. This means that $a_1 b_1 + a_2 b_2 - a_1 b_2 - a_2 b_1 \geq 0$, so $a_1 b_1 + a_2 b_2 \geq a_1 b_2 + a_2 b_1$.

To get equality, we need $xy = 0$. Then, either $x = 0$, so $a_1 = a_2$ or $y = 0$, meaning $b_1 = b_2$.

(b) Let us look at what happens when a_1 is not paired with b_1. Then a_1 is paired with b_i and b_1 is paired with a_j. We know that $a_1 \leq a_j$ and $b_1 \leq b_i$.

But then, the first part tells us that $a_1 b_1 + a_i b_j \geq a_1 b_i + a_j b_1$. This means that we can make our sum of products greater (or remain the same) by swapping the pairings to make a_1 and b_1 pair together.

We can repeat this to make a_2 pair with b_2, and a_3 pair with b_3. This means the maximal value is $a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$, with the sequences in the same order.

(c) Let us consider the right hand side of this equation. We can write this as

$$\left(\frac{c_1+c_2+c_3+c_4}{4}\right)^2 = \frac{1}{16}\left(c_1^2+2c_1c_2+c_2^2+2c_1c_3+2c_2c_3+c_3^2+2c_1c_4+2c_2c_4+2c_3c_4+c_4^2\right).$$

Then, we can group this right hand side into the sum of four products, so that:

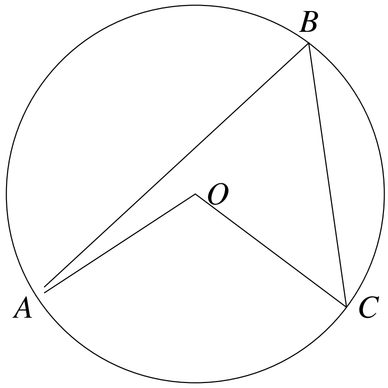
$$\begin{aligned}\left(\frac{c_1+c_2+c_3+c_4}{4}\right)^2 = &\frac{1}{16}\left(c_1^2+c_2^2+c_3^2+c_4^2\right)\\ &+\frac{1}{16}(c_1c_2+c_2c_3+c_3c_4+c_4c_1)\\ &+\frac{1}{16}(c_1c_3+c_2c_4+c_3c_1+c_4c_2)\\ &+\frac{1}{16}(c_1c_4+c_2c_1+c_3c_2+c_4c_3).\end{aligned}$$

Now, we can consider applying the second part to each of these factors. We're taking $a_i = b_i = c_i$, for which the maximum is $c_1^2+c_2^2+c_3^2+c_4^2$. This means:

$$\begin{aligned}\left(\frac{c_1+c_2+c_3+c_4}{4}\right)^2 &\leq \frac{4}{16}\left(c_1^2+c_2^2+c_3^2+c_4^2\right)\\ &= \frac{c_1^2+c_2^2+c_3^2+c_4^2}{4}.\end{aligned}$$

This is precisely what we wanted.

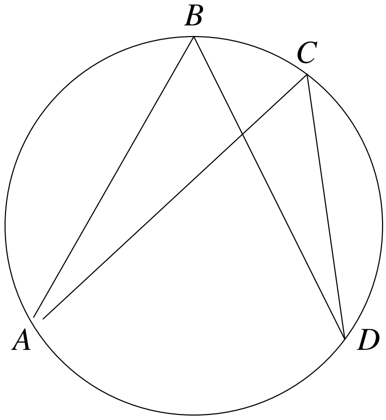
4. (a) Suppose that A, B and C lie on a circle with centre O.



Prove that $\angle AOC = 2\angle ABC$.

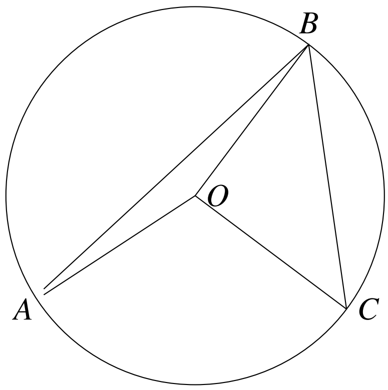
This is one of the circle theorems, which you may have come across before. Here, however, you should prove it without using any of the circle theorems.

(b) Suppose that A, B, C and D, in that order, all lie on the same circle . Prove that $\angle ABD = \angle ACD$.



Solution

(a) Let us draw in the line OB.



Since $OA = OB$ (they are both radii of the same circle), we know that triangle AOB is isosceles. This means that $\angle BAO = \angle ABO$. Since angles in a triangle add up to 180°, we know that $\angle AOB = 180° - \angle ABO - \angle OAB = 180° - 2\angle ABO$.

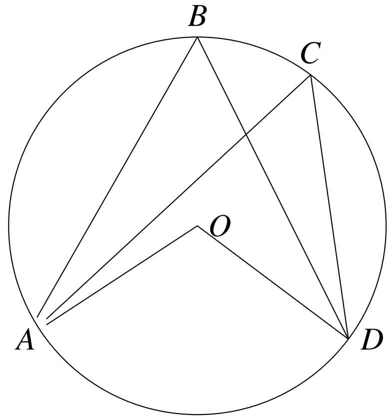
Using the same argument, we know that $\angle BOC = 180° - 2\angle OBC$. Then, using the fact that angles at

a point add up to 360°, we get:

$$\begin{aligned}\angle AOC &= 360^\circ - \angle BOA - \angle COB \\ &= 360^\circ - (180^\circ - 2\angle ABO) - (180^\circ - 2\angle OBC) \\ &= 2\angle ABO + 2\angle OBC \\ &= 2\angle ABC\end{aligned}$$

This is exactly what we wanted to prove.

(b) We can use the first part of the question to help us, by drawing in the centre of the circle.



Applying the first part of the question twice, we get $2\angle ABD = \angle AOD = 2\angle ACD$. This tells us exactly that $\angle ABD = \angle ACD$.

5. A *Pythagorean triple* is a triple of integers (a, b, c) such that $a^2 + b^2 = c^2$.

(a) i. Show that all square numbers are congruent to either 0 or 1 modulo 3.

ii. Hence show that a or b (or both) must be divisible by 3.

(b) Show that at least one of a, b and c must be divisible by 5.

(c) Show that a or b (or both) must be divisible by 4.

(d) Could there be any other divisibility results of this type?

Solution

(a) i. Working modulo 3, we need only to consider the squares of 0, 1 and 2, since every number is congruent to one of these modulo three.

Then, $0^2 \equiv 0 \pmod 3$, $1^2 \equiv 1 \pmod 3$ and $2^2 = 4 \equiv 1 \pmod 3$. This tells us that these are the only possible values of squares modulo 3.

ii. We know that each of a^2, b^2 and c^2 must be congruent to 0 or 1 modulo 3. This means that a^2 and b^2 cannot both be congruent to 1, as then their sum would be $2 \pmod 3$, which is not a square. But, we saw in the first part that if $x^2 \equiv 0 \pmod 3$, then $x \equiv 0 \pmod 3$, so one of a and b must be divisible by 3.

(b) Let us consider the squares modulo 5. We only need to consider the squares of 0, 1, 2, 3 and 4, as all

other numbers are congruent to one of these.

$$\begin{aligned}
0^2 &= 0 \equiv 0 \pmod 5. \\
1^2 &= 1 \equiv 1 \pmod 5. \\
2^2 &= 4 \equiv 4 \pmod 5. \\
3^2 &= 9 \equiv 4 \pmod 5. \\
4^2 &= 16 \equiv 1 \pmod 5.
\end{aligned}$$

This tells us that all the squares are congruent to 0, 1 or 4 modulo 5.

Then, we can think about what happens if neither a^2 nor b^2 is congruent to 0. The possible values of $a^2 + b^2$ are $1 + 1 \equiv 2 \pmod 5$, $1 + 4 \equiv 0 \pmod 5$ and $4 + 4 \equiv 3 \pmod 5$. The only way in which this can be a square is in the middle case, so $c^2 \equiv 0 \pmod 5$.

This tells us that one of a^2, b^2 and c^2 must be congruent to 0 modulo 5. But the list above tells us that this means one of a, b and c must be divisible by 5.

(c)

> This part of the question needs you to choose the correct modulus to work in. Since we are looking for a factor of 4, which is a power of 2, powers of 2 are good choices to choose.
>
> If you try to work modulo 4, this tells you only that one of a, b or c must be even. Working modulo 8, you can deduce that one of a or b or c must be divisible by 4. The next case is working modulo 16.

Let us consider the squares modulo 16:

$$\begin{aligned}
0^2 \equiv 4^2 \equiv 8^2 \equiv 12^2 &\equiv 0 \pmod{16}; \\
1^2 \equiv 7^2 \equiv 9^2 \equiv 15^2 &\equiv 1 \pmod{16}; \\
2^2 \equiv 6^2 \equiv 10^2 \equiv 14^2 &\equiv 4 \pmod{16}; \\
3^2 \equiv 5^2 \equiv 11^2 \equiv 13^2 &\equiv 9 \pmod{16}.
\end{aligned}$$

Then, it is worth noticing that a or b being divisible by 4 is equivalent to their squares being divisible by 16, as $16 = 4^2$. This means we need to consider only the other possibilities for a and b. The possibilities are:

$$\begin{aligned}
1 + 1 &\equiv 2 \pmod{16}; \\
1 + 4 &\equiv 5 \pmod{16}; \\
1 + 9 &\equiv 10 \pmod{16}; \\
4 + 4 &\equiv 8 \pmod{16}; \\
4 + 9 &\equiv 13 \pmod{16}; \\
9 + 9 &\equiv 2 \pmod{16}.
\end{aligned}$$

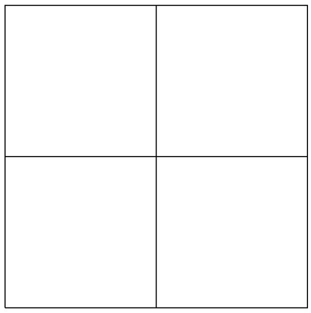
None of these are squares, so we know that we cannot get a Pythagorean triple without either a or b being divisible by 4.

(d) We can look at the triple $3^2 + 4^2 = 5^2$. This means that no other numbers (apart from 2) must divide one of a, b or c.

6. Suppose that five points lie inside a square of side length 2. Show that some pair of them must be at a distance of at most $\sqrt{2}$ from each other.

SOLUTION

> In the last sheet, you met the pigeonhole principle, which said that if there are $k + 1$ pigeons to fit into k pigeonholes, then at least one pigeonhole must have more than one pigeon in it. We will apply this to this question.



The square can be divided up into four smaller squares, each of which have side length 1. Since there are five points and four smaller squares, the pigeonhole principle tells us that one of these squares must contain two of the five points.

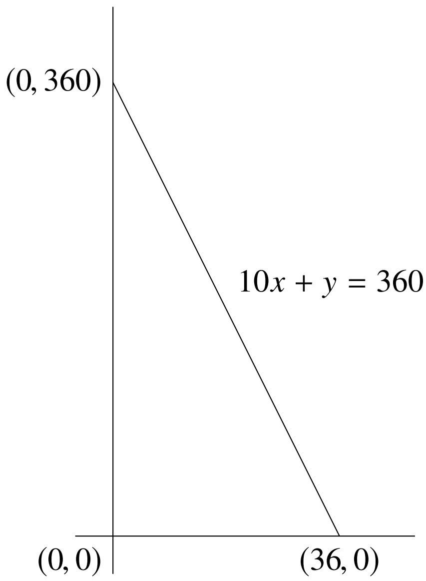
The furthest apart that two points can be in the same small square is at opposite ends of the diagonal. This has length, by Pythagoras' theorem, $\sqrt{1^2 + 1^2} = \sqrt{2}$, so these two points are at distance at most $\sqrt{2}$ from each other.

7. Let O be the origin. How many ways are there of choosing points P and Q, with non-negative integer coordinates and lying on the line $10x + y = 360$, such that the triangle OPQ has integer area?

ANSWER 1332

SOLUTION

We can start by drawing a useful diagram.



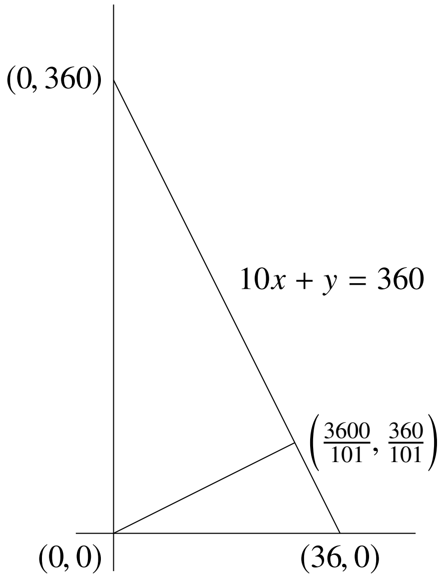
Then, we want to think of our triangles as having their bases on the line $10x + y = 360$. All such triangles which have their third vertex at $(0, 0)$ will have the same perpendicular height.

To calculate the perpendicular height, we need to find the line through $(0, 0)$ perpendicular to $10x+y = 360$. We can re-write $10x+y = 360$ as $y = 360-10x$, which has gradient -10. This means that the perpendicular line must have gradient $\frac{1}{10}$. Since it passes though $(0, 0)$, it must have equation $y = \frac{x}{10}$.

Then the intersection of the line and its perpendicular occurs where both equations are satisfied, that is

$$\begin{aligned} 10x + y &= 360; \\ y &= \frac{x}{10}. \end{aligned}$$

Substituting the second equation into the first gives $\frac{101}{10}x = 360$, or $x = \frac{3600}{101}$. Then, $y = \frac{360}{101}$.



Now, using Pythagoras' theorem, we can calculate these perpendicular heights to be

$$\sqrt{\left(\frac{3600}{101}\right)^2 + \left(\frac{360}{101}\right)^2} = \frac{360}{101}\sqrt{10^2 + 1^2} = \frac{360}{\sqrt{101}}.$$

We are now ready to choose the two points P and Q. It is worth noticing that the y-coordinate is always an integer when the x coordinate is, as $y = 360 - 10x$. Therefore we only need to consider the choice of x-coordinate.

If we take two choices, say x_P and x_Q, then these have y-coordinates of $360 - 10x_P$ and $360 - 10x_Q$ respectively. This means that the distance between them is

$$\begin{aligned} \sqrt{(x_P - x_Q)^2 + \left((360 - 10x_P) - (360 - 10x_Q)\right)^2} &= \sqrt{(x_P - x_Q)^2 + (10x_Q - 10x_P)^2} \\ &= \sqrt{101(x_P - x_Q)^2} \\ &= |x_P - x_Q|\sqrt{101}. \end{aligned}$$

This means that, for any choice of x_P and x_Q, the area of the triangle formed will be

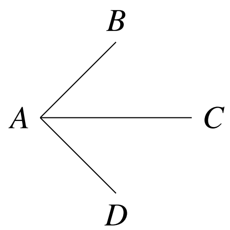
$$\frac{1}{2} \times \frac{360}{\sqrt{101}} \times |x_P - x_Q|\sqrt{101} = 180|x_P - x_Q|.$$

Hence, for any permitted choice of x_P and x_Q, the area will be an integer. There are 37 choices for x_P (any integer between 0 and 36) and then 36 choices for x_Q, as they need to be different. This gives a total of $36 \times 37 = 1332$ different triangles.

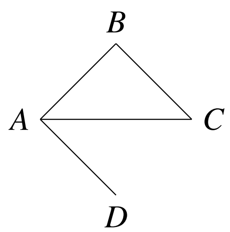
8. Suppose there are six students, each pair of whom either know each other or do not know each other. Prove that either there are three students who all know each other or three who are all strangers.

SOLUTION

Let us consider one of the students A, say. Suppose that she knows at least three of the others, say B, C and D. We can draw a diagram to represent this. (In the diagrams that follow, a solid line represents knowing someone and a dashed line represents not knowing them.)

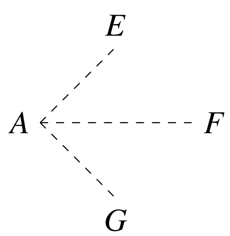


Then, if any of B, C and D know each other, this means that they, together with A would form three students who all know each other, e.g.



Otherwise, B, C and D would all not know each other, which is also what we wanted.

We supposed at the beginning that A knew at least three of the students. So what happens if she did not? Then she would know at most two of the others, so would not know at least three of the six, say E, F and G.



If any pair of E, F and G do not know each other, then these two and A form a set of three students, all of whom do not know each other. Otherwise, we have that E, F and G all know each other, which is what we wanted.

United Kingdom
Mathematics Trust

Mentoring Scheme

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Julia Robinson

Sheet 3

Solutions and comments

This programme of the Mentoring Scheme is named after Julia Robinson (1919–1985).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Robinson_Julia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

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> **1.** I do not realise there was a power-cut during the night, which has caused the electric clock to pause for the duration of the outage. I leave to walk to the station as usual, when the electric clock (which is usually correct) says 8 o'clock. So I am surprised when I get to the station and the clock there says 9:30, which the ticket inspector assures me is the correct time. I catch the next train. When I return in the evening the station clock says 7:30 as I leave to walk home. Of course, I am tired now, so I can only walk at two-thirds of the pace I managed in the morning. When I get home, the hall clock says 7 o'clock. Assuming there has not been another power cut during the day, how long was the power off last night?

Solution

> The key to this question is to translate the problem into algebra. There are two key times that the question does not tell us: the time that it takes the narrator to walk to the station and the time that the electricity was off overnight. Thus it makes sense to give both of these variables names.

Let t be the amount of time that the electricity was off overnight and w the amount of time that it takes the narrator to walk to the station. Let us measure both these in minutes.

In the morning, the narrator set off to work at t minutes after $8:00$ and arrived at $9:30$. Hence the time to walk to the station was $90 - t$ minutes and $w = 90 - t$.

In the evening, the narrator set off from the station at $19:30$ and arrived home at t minutes after $19:00$, so the journey took $t - 30$ minutes. However, at $\frac{2}{3}$ of the speed, the journey would take $\frac{3}{2}$ of the time that it did in the morning. (This is because the distance is the same, so the speed and time are inversely proportional.) Therefore $t - 30 = \frac{3}{2}w$.

This gives the following pair of simultaneous equations:

$$\begin{aligned} 90 - t &= w; \\ t - 30 &= \tfrac{3}{2}w. \end{aligned}$$

We can add these equations together to tell us that

$$60 = \tfrac{5}{2}w.$$

Then, multiplying both sides by $\frac{2}{5}$, we get

$$w = 24.$$

Substituting this into the first equation yields

$$90 - t = 24.$$

Therefore,

$$t = 90 - 24 = 66.$$

This means that the power was off for 66 minutes.

2. (a) Every child in a class plays either hockey or chess. The register shows that 24 students play hockey (some of whom also play chess); 14 students play chess (some of whom also play hockey). If 7 students play both, then how many students are there in the class?

(b) Suppose that A and B are finite sets. Show that $|A \cup B| = |A| + |B| - |A \cap B|$. This is called the inclusion-exclusion principle.

For a set X, $|X|$ denotes the size of X, or the number of elements in X. $X \cap Y$, or X intersect Y, denotes the set of elements that are in both X and Y. $X \cup Y$, or X union Y, denotes the set of elements that are in at least one of X or Y.

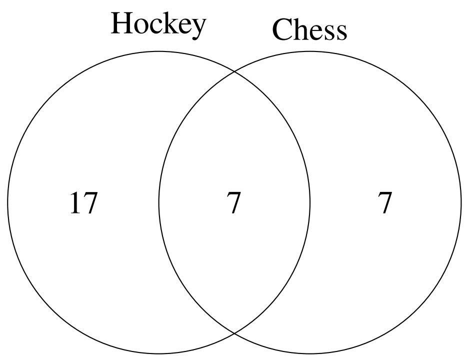
Answer

(a) 31

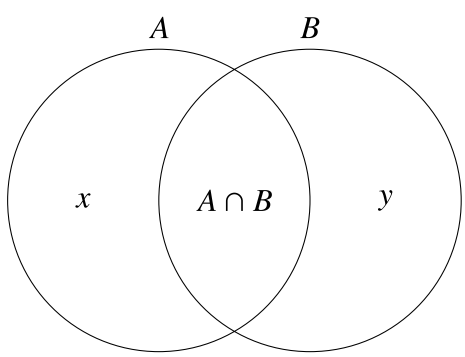
Solution

(a) If there are 24 students who like hockey and 7 students who like both, then there are 17 students who like only hockey. Likewise, if there are 14 students who like chess, of whom 7 like hockey, then there are 7 who like only chess.

This means there is a total of $17 + 7 + 7 = 31$ students in the class. We can visualise this on a Venn diagram.



(b) Let us consider $|A| + |B|$. The Venn diagram below shows the three regions x, $A \cap B$ and y. (Here x consists of of the elements of A not in $A \cap B$ and y is defined similarly.)



This diagram shows that we have all the regions in $A \cup B$, with an extra copy of $A \cap B$. This means $|A| + |B| = |A \cup B| - |A \cap B|$, which is exactly the result that we wanted.

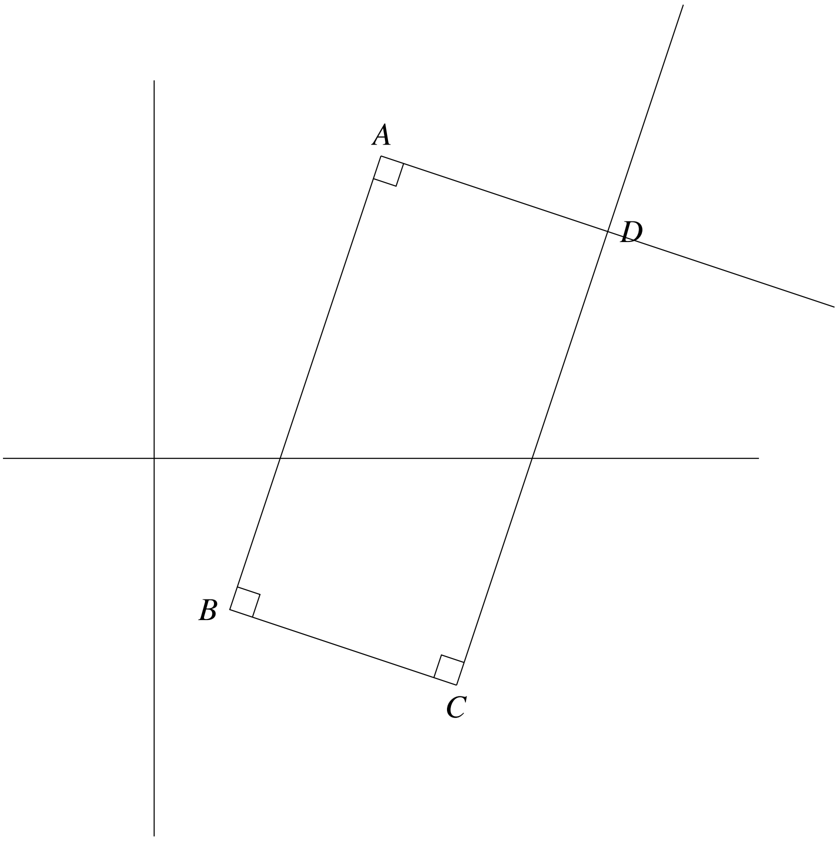
3. The points $A\ (3, 4)$, $B\ (1, k)$ and $C\ (4, -3)$ are three vertices of a rectangle $ABCD$. Find all possible values of k. *Note that the rectangle is labelled working cyclically around the rectangle as usual.*

Answer $k = -2$, $k = 3$

Solution

> In this question it is worth considering the exact properties A, B and C need to have in order to form a rectangle with a fourth point. To do this, we need to use a suitable definition of a rectangle. (There is some choice here.)
>
> For this solution, I am going to define a rectangle as a quadrilateral with four right angles. You might want to think about proving that this is the same as some of the other definitions.

Since we are assuming that all four angles in a rectangle are right angles, this means that a value of k that produces a rectangle must both make $\angle ABC$ a right angle and must allow D to be placed in such a way as to allow the other three angles to be right angled.



For any given location of B with $\angle ABC = 90°$, we can draw the perpendiculars to the lines BA and BC at A and C respectively. Then we can put D where these lines meet, as in the diagram. We have three right angles in quadrilateral $ABCD$, so, as the sum of all four angles is $360°$, we have $\angle CDA = 360° - 3 \times 90° = 90°$. Then $ABCD$ has four right angles, so it is a rectangle.

Therefore, all we need to do to ensure we can have a rectangle is to make sure that $\angle ABC$ is a right angle, which is the same as saying that the lines AB and BC are perpendicular. These lines are perpendicular exactly when their gradients multiply to -1. The gradients are the change in the y direction divided by the change in the x direction, which are:

$$\text{for } AB, \quad \frac{k-4}{1-3} = \frac{4-k}{2};$$
$$\text{for } BC, \quad \frac{-3-k}{4-1} = -\frac{k+3}{3}.$$

The product of these must be -1, so we know that

$$-\frac{(4-k)(k+3)}{6} = -1.$$

Multiplying both sides of the equation by 6 gives

$$(k-4)(k+3) = -6.$$

Expanding the brackets and collecting like terms, we get

$$k^2 - k - 6 = 0.$$

Now we can factorise this equation to obtain

$$(k-3)(k+2) = 0.$$

Therefore, $k = 3$ or $k = -2$.

We then need to check that these are valid solutions that make the product of the gradients equal to -1. This will ensure that we do get a right angle.

- In the case $k = -2$, AB has gradient $\frac{-6}{-2} = 3$ and BC has gradient $\frac{-1}{3}$, which multiply together to give -1, so this is a valid solution.
- In the case $k = 3$, AB has gradient $\frac{1}{2}$ and BC has gradient $-\frac{6}{3} = -2$, which also multiply together to give -1, so this is another valid solution.

4. Solve the system of equations

$$\begin{aligned} x^2 - xy + y^2 &= 3 \\ x^2 - y^2 &= 3 \end{aligned}$$

ANSWER $(x, y) = \left(\sqrt{3}, 0\right), \left(-\sqrt{3}, 0\right), (2, 1)$ and $(-2, -1)$

SOLUTION

> It is not obvious how to tackle this question. A good way to start is to try playing with the equations, to see if you can get anything helpful. For example, you might be able to get some of the terms to cancel.

Let's subtract the second equation from the first, as then both the x^2 and constant (the 3) terms cancel. This gives

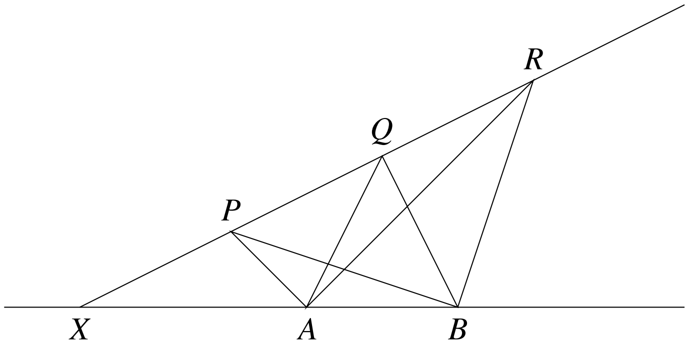
$$2y^2 - xy = 0.$$

Then all the terms in this equation have a factor of y, so we can write it as $y(2y - x) = 0$, implying that $y = 0$ or $x = 2y$.

In the first case, substituting into the original equations gives $x^2 = 3$, so $x = \pm\sqrt{3}$. In the second case, performing the substitution gives $3y^2 = 3$, so $y = \pm 1$.

This gives us four candidate solutions for the pair (x, y): $\left(\sqrt{3}, 0\right), \left(-\sqrt{3}, 0\right), (2, 1)$ and $(-2, -1)$. For each solution, we must verify that it is valid by substituting it into the two original equations.

5. (a) Which of these triangles ABP, ABQ and ABR has the largest area? How do you know?



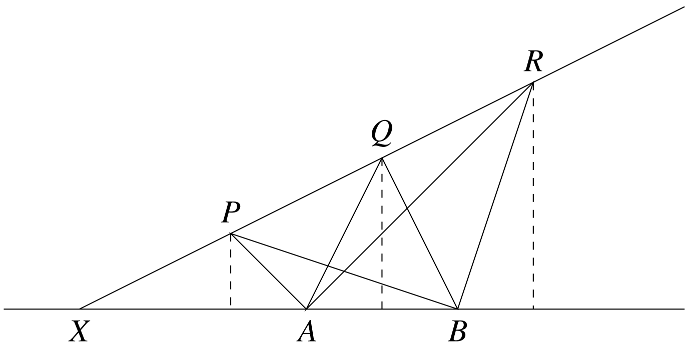
(b) Given a square of side length 1, what is the area of the largest triangle that you can draw inside it?

Answer

(a) ABR

(b) $\frac{1}{2}$

Solution



(a)

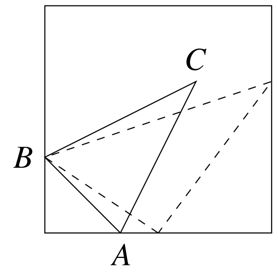
As each of the triangles has the same base AB, we can consider the perpendicular height of each of the triangles: whichever is largest will correspond to the largest area.

The further you move away from X, the larger the distance from the line XB. This means that the largest perpendicular height corresponds to the furthest distance from X, which means triangle ABR has the largest area.

(b)

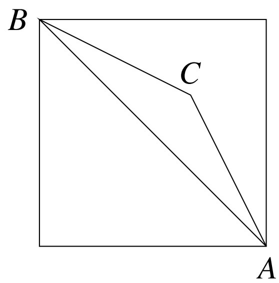
> There are lots of things you can think of to try and find the largest triangle. If you want to consider some different triangles, then you might decide to play with taking a triangle and deforming it to make it bigger. There are other approaches you can take, but that is the approach I will try here. You might like to try and think of some other methods.

Let us start with a triangle. What happens if none of its vertices lie on a particular edge of the square? Then we could imagine stretching the triangle until one of its vertices does lie on an edge of the square. Since we are stretching with a scale factor of at least 1, this will increase the area.



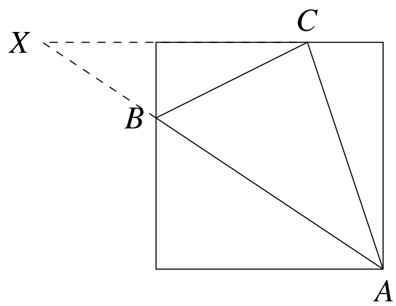
If needed, we can do this in any direction: then at least one triangle vertex is on each side of the square. There are three triangle vertices and four sides to the square, so at least one vertex is on two sides, and therefore is in the corner.

Then there are two possibilities. Either the other two triangle vertices are each on a side of the square, or one of them is in the opposite corner. Let us first consider the case where we have two vertices of the triangle in opposite corners of the square.



If we fix the line AB between the opposite corners, then we want to move C to make the area as large as possible. This means making the perpendicular height away from AB as large as possible. This in turn occurs when C in either of the other corners. The resulting triangle has area $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$.

Otherwise, we have two of the vertices on different edges and the third in a corner.



Now, consider fixing the locations of A and B and moving C to maximise the area. We can see that this will again occur with the vertex in a corner of the square, adjacent to the current corner (A) we know about. The diagram shows that we cannot move C the opposite way, since the line AB and the side containing C meet outside the square. This then gives us a triangle with two vertices in adjacent corners of the square and the final vertex on the opposite side. The adjacent corners form the base, and the area is $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$.

What we have done here is taken any triangle in the square and performed a series of operations on it. None of these made the area smaller. We have ended up with an area of $\frac{1}{2}$, so we must have started with an area less than or equal to $\frac{1}{2}$. Therefore the largest possible area is $\frac{1}{2}$.

6. How many ways are there to choose positive integers a, b, c and d such that $a < b < c < d$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ is an integer?

Answer 7

Solution

The largest possible value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$ is $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=2\frac{1}{12}$, so the only possible integer totals are 1 and 2.

If $a \geq 3$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \leq \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{19}{20}<1$, so the total cannot be an integer. This means that we must have either $a=1$ or $a=2$

If $a=1$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} > \frac{1}{a}=1$, so the total must be 2. If $b \geq 3$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \leq \frac{1}{1}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=1\frac{47}{60}<2$, so there are no integer solutions. As $b>a$, we must have $b=2$.

Then, if $c \geq 4$, we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \leq \frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}=1\frac{19}{20}<2$, so no solutions. Therefore $c=3$. Then, $\frac{1}{d}=2-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}=2-\frac{1}{1}-\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$, so $d=6$, and we have one solution of $a=1$, $b=2$, $c=3$ and $d=6$.

Otherwise, $a=2$. Then, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \leq \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=1\frac{17}{60}<2$, so the only possible integer total is 1. Then, if $b \geq 6$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \leq \frac{1}{2}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}=\frac{154}{168}<1$, so there are no solutions. Therefore we have either $b=3$, $b=4$ or $b=5$, as $b>a$ also.

> It is possible to continue using this type of bounding argument, which will produce a perfectly good solution. In this next part though, there is a neat factorisation that allows us to count quickly how many solutions there are.

- If $b=3$, then $\frac{1}{a}+\frac{1}{b}=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, so $\frac{1}{c}+\frac{1}{d}=1-\frac{5}{6}=\frac{1}{6}$.

  We can rearrange this equation to give $cd-6c-6d=0$. Then this is equivalent to $(c-6)(d-6)=36$. The possible factorisations of 36 are 1×36, 2×18, 3×12, 4×9, 6×6, -1×-36, -2×-18, -3×-12, -4×-9 and -6×-6. Since $3=b<c<d$, this means that $c-6$ can only be 1, 2, 3 or 4, as it must be the smaller factor. This adds four solutions.

- If $b=4$, then $\frac{1}{a}+\frac{1}{b}=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$, so $\frac{1}{c}+\frac{1}{d}=1-\frac{3}{4}=\frac{1}{4}$.

  We can rearrange this equation to give $cd-4c-4d=0$. Then, this is equivalent to $(c-4)(d-4)=16$. The possible factorisations of 16 are 1×16, 2×8, 4×4, -1×-16, -2×-8 and -4×-4. We know that $4=b<c<d$, so $0<c-4<d-4$. This means $c-4$ can only be 1 or 2, so $c=5$ and $d=20$ or $c=6$ and $d=12$. This adds another 2 solutions.

- If $b=5$, then $\frac{1}{a}+\frac{1}{b}=\frac{1}{2}+\frac{1}{5}=\frac{7}{10}$, so $\frac{1}{c}+\frac{1}{d}=\frac{3}{10}$, which rearranges to give $9cd-30c-30d=0$.

  > We have multiplied this by 3 to make sure that the coefficient of cd is a square number, as that allows us to factorise as we want to.

  Now, we can factorise this as $(3c-10)(3d-10)=100$. The factorisations of 100 are 1×100, 2×50, 4×25, 5×20, 10×10, -1×-100, -2×-50, -4×-25, -5×-20 and -10×-10. As $6\leq c<d$, we have $8\leq 3c-10<3d-10$. However, we can see that none of our factorisations satisfy this: the only possibility with both factors at least 8 is 10×10, but this has the factors equal, so there are no extra solutions.

Therefore there is a total of seven solutions, with (a,b,c,d) being $(1,2,3,6)$, $(2,3,7,42)$, $(2,3,8,24)$, $(2,3,9,18)$, $(2,3,10,15)$, $(2,4,5,20)$ and $(2,4,6,12)$.

7. Let a, b and c be positive integers such that $\frac{a\sqrt{2}+b}{b\sqrt{2}+c}$ is a rational number. Prove that $a + b + c$ is a factor of $a^2 + b^2 + c^2$.

Solution

We know that the given quantity is rational, so we can write it as a fraction with p and q integers:

$$\frac{a\sqrt{2} + b}{b\sqrt{2} + c} = \frac{p}{q}$$

We can make sure that we write this fraction in its simplest form, assuming that p and q have no common factors larger than 1.

> It is not immediately obvious why we want these numbers to be coprime but it helps us later. It certainly is not what I thought initially. Later on in the proof we will find it useful: I thought about it at that point and looked back to see if it could be true.

If we multiply by the denominators, we then get the following equation

$$aq\sqrt{2} + bq = pb\sqrt{2} + pc.$$

We can rearrange this to give

$$(aq - pb)\sqrt{2} = pc - bq.$$

Now, we know that a, b, c, p and q are integers, so $cp - bq$ and $aq - bp$ are integers, and unless $aq - bp = 0$, we could write $\sqrt{2} = \frac{cp-bq}{aq-bp}$. However, this is impossible as $\sqrt{2}$ is irrational, so we must have:

$$cp = bq;$$
$$aq = bp.$$

Now, we can use the fact that p and q are coprime to tell us something about b. The first equation implies that p is a factor of bq. But p and q are coprime, so it follows that p divides b. Likewise, the second equation tells us that q divides b.

The previous two results tell us that the lowest common multiple of p and q must divide b. Now we can again use the fact that p and q are coprime to deduce that the lowest common multiple is pq, and therefore we can write $b = xpq$, for some integer x.

Substituting this into the two equations above, we get that $a = xp^2$ and $c = xq^2$. (To do this we have to divide by each of p and q. The original expression for $\frac{p}{q}$ was positive, as a, b and c were, so we know neither p nor q are 0). Then, we can write:

$$a + b + c = x\left(p^2 + pq + q^2\right);$$
$$a^2 + b^2 + c^2 = x^2\left(p^4 + p^2q^2 + q^4\right).$$

We would like to show that the top expression is a factor of the bottom expression. Certainly x is a factor of x^2, so we could achieve our goal if we showed that $p^2 + pq + q^2$ divides $p^4 + p^2q^2 + q^4$. We need terms p^4 and q^4, which suggests multiplying by $p^2 + q^2$. This gives

$$\left(p^2 + q^2\right)\left(p^2 + pq + q^2\right) = p^4 + p^3q + 2p^2q^2 + pq^3 + q^4 = \left(p^4 + p^2q^2 + q^4\right) + \left(p^3q + p^2q^2 + pq^3\right).$$

This extra term is exactly pq times $p^2 + pq + q^2$, so

$$\left(p^2 - pq + q^2\right)\left(p^2 + pq + q^2\right) = p^4 + p^2q^2 + q^4.$$

This means we can write

$$a^2 + b^2 + c^2 = p^4 + p^2q^2 + q^4 = x^2\left(p^2 - pq + q^2\right)\left(p^2 + pq + q^2\right) = x\left(p^2 - pq + q^2\right)(a + b + c).$$

Therefore $a + b + c$ is a factor of $a^2 + b^2 + c^2$, as required.

8. Sophie and Tom are playing a game. They take it in turn to write down a digit in a number, but they are allowed to use the digits 1, 2, 3, 4 and 5 only. Sophie writes down the first digit.

(a) Suppose that they are writing down a total of eighteen digits. Tom wins only if the final eighteen-digit number is divisible by 3. If both players use a perfect strategy, then who should win?

(b) What if Tom wins only if the final eighteen-digit number is divisible by 9?

(c) What could happen if they went on to a twenty-digit number rather than one with eighteen digits? [Tom still wins if the final number is divisible by 9.]

Answer

(a) Tom
(b) Tom
(c) Sophie

Solution

(a) Tom wants to ensure that the total is a multiple of 3 after the final turn. We know that a number is a multiple of 3 exactly when the sum of the digits is a multiple of 3. Therefore Tom wants to make the sum of the digits divisible by 3.

One way in which he can do this is to ensure that, after each of his turns, the sum of the digits is divisible by 3. He can do this by choosing the digit $6 - s$ after Sophie chooses the digit s. This means that the sum of the digits after each of Tom's turns increases by 6, so is always a multiple of 3.

If Tom uses this strategy, then the sum of the digits will be $6 \times 9 = 54$. This number is a multiple of 3 and so Tom will win.

(b) We know that a number is divisible by 9 exactly when the sum of the digits is divisible by 9. But then, Tom can use the same strategy as in the first part to ensure that the total is 54, which is divisible by 9.

(c) Notice that Tom chooses the final digit. For him to win, he needs to make the sum of the digits divisible by 9. He can do this as long as the sum of the digits when Sophie has written down the final one has remainder 4, 5, 6, 7 or 8 when divided by 9.

Therefore, for Sophie to win she needs to ensure that the remainder when the sum of the digits after she has placed her final digit is 0, 1, 2 or 3.

To achieve this, Sophie can start by writing down either a 1, 2 or 3. Then, she can use the same strategy as Tom used in the previous part to ensure that the sum of the next eighteen digits is divisible by 9. Notice that she is essentially going second amongst this set of eighteen numbers, so she can use the strategy. In this way, the total before Tom's final move has remainder 1, 2 or 3 when divided by 9, so Tom has no way to make the total divisible by 9. Therefore Sophie can guarantee a win.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Julia Robinson

Sheet 4

Solutions and comments

This programme of the Mentoring Scheme is named after Julia Robinson (1919–1985).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Robinson_Julia.html for more information.

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1. The terms in the sequence $9, 17, 24, 36 \ldots$ are formed from the sum of the corresponding terms in two other sequences. In the first sequence, each term is the sum of the two preceding terms. The second sequence is an arithmetic sequence, where consecutive terms have a constant difference.

What is the seventh term of the resulting sequence?

Answer 115

Solution

In the first unknown sequence, each term is the sum of the previous two terms. Therefore, if the first two terms are a and b, then the sequence becomes

$$a, b, a + b, a + 2b, \ldots$$

In the second sequence, if the first term is c, and the difference between consecutive terms is r, then the sequence is

$$c, c + r, c + 2r, c + 3r, \ldots$$

Therefore, since the original sequence is the sum of these terms we have the following equations:

$$\begin{aligned} a + c &= 9; \\ b + c + r &= 17; \\ a + b + c + 2r &= 24; \\ a + 2b + c + 3r &= 36. \end{aligned}$$

Adding the second and third of these equations yields $a + 2b + 2c + 3r = 41$.

Subtracting the fourth equation, we get $c = 41 - 36 = 5$.

Then, from the first equation $a + c = 9$, $a = 4$.

Substituting these values into the second and third of the equations reduces them to:

$$\begin{aligned} b + r &= 12; \\ b + 2r &= 15. \end{aligned}$$

Subtracting again, we get $r = 3$ and $b = 9$. This tells us that the original sequences were:

$$4, 9, 13, 22, \ldots;$$
$$5, 8, 11, 14, \ldots.$$

By adding these, we can check that we get the correct resulting sequence. We can then extend the two sequences above to their seventh terms to obtain:

$$4, 9, 13, 22, 35, 57, 92, \ldots;$$
$$5, 8, 11, 14, 17, 20, 23 \ldots.$$

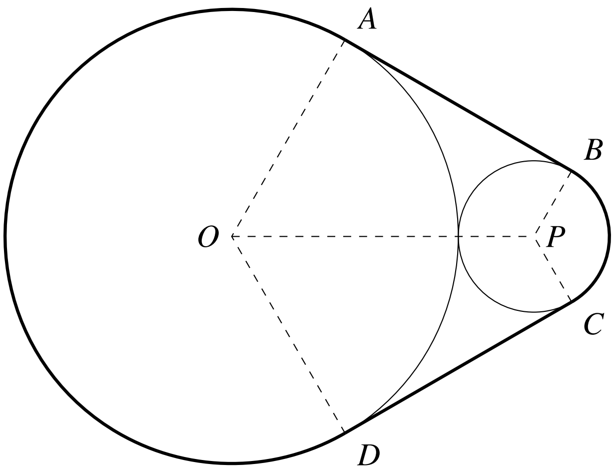
Therefore the seventh term of the original sequence is $92 + 23 = 115$.

2. Two cylinders have radii 4cm and 12cm respectively. What is the length of the shortest band that can be fitted around both cylinders to hold them together?

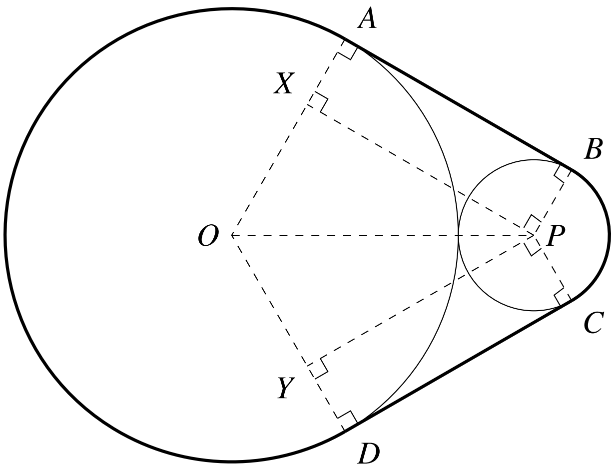
Answer $18\frac{2}{3}\pi + 16\sqrt{3}$

Solution

Let us consider a cross-section of the cylinders.



Let O and P be the centres of the cylinders. Then, $OA = OD = 12$ and $PB = PC = 4$. $OP = 3 + 1 = 4$. Since the band fits as tightly as possible, this tells us that the lines AB and CD must be common tangents of the two circles. This tells us that $\angle OAB = \angle ABP = \angle ODC = \angle DCP = 90°$.



If X is located on OA such that PX is perpendicular to OA, and likewise for Y on OB, then we have that $ABPX$ must be a rectangle, as each of its angles must be $90°$.

This tells us that $AX = BP = 4$ and $AB = PX$. Therefore $OX = OA - XA = 12 - 4 = 8$.

We can apply Pythagoras' theorem to triangle OXP to tell us that $XP = \sqrt{OP^2 - OX^2} = \sqrt{16^2 - 8^2} = \sqrt{192} = 8\sqrt{3}$. Hence $AB = 8\sqrt{3}$. The same argument tells us that $CD = 8\sqrt{3}$.

We would also like to know $\angle DOA$, as then we could work out the length of band on the cylinder. This angle is equal to $2\angle POX$, using symmetry. But, we know that $\angle PXO = 90°$ and $PO = 16 = 2OX$. This means that triangle POX is half of an equilateral triangle and hence $\angle POX = 60°$.

It follows that, on the large circle, the band stretches for $360° - 2 \times 60° = 240°$. The corresponding length

is

$$\frac{240^\circ}{360^\circ} \times 2\pi \times 12 = 16\pi.$$

On the small circle, the minor arc BC is of length

$$\frac{120^\circ}{360^\circ} \times 2\pi \times 4 = \frac{8\pi}{3}.$$

Therefore the total length of the band is $16\pi + 8\sqrt{3} + \frac{8\pi}{3} + 8\sqrt{3} = 18\frac{2}{3}\pi + 16\sqrt{3}$.

> **3.** Is it possible to number the eight vertices of a cube with all the integers from 1 to 8 in such a way that the sums of the numbers at the ends of each edge are different?

Answer No

Solution

The smallest possible sum of the numbers at the end of an edge is $1 + 2 = 3$. The largest possible sum is $7 + 8 = 15$. Therefore there are 13 possible sums.

> A cube has 12 edges, so if each edge is to have a unique sum, then all but one of these values must be used. An important observation you may make when trying to construct an example is that there are some sets of numbers that you cannot place together on the edges.

We are now going to show that you cannot get all the sums 12, 13, 14 and 15 on the edges. Suppose all of 15, 14 and 13 are on edges. The only way of making 15 is $7 + 8$, and the only way of making 14 is $8 + 6$. This means that 7 and 6 are on opposite corners of the same face, and therefore are not adjacent. This means that 13 cannot be made as $7 + 6$, so must be $8 + 5$. This further means that 5 is also adjacent to 8 and therefore not adjacent to 7.

Now, the only ways to obtain a sum of 12 are $8 + 4$ and $7 + 5$. We know that 7 and 5 are not adjacent, since they are both adjacent to the 8 edge. However, we also know that 8 is adjacent to 7, 6 and 5, and therefore cannot be adjacent to any other numbers. Therefore we cannot make a sum of 12. This means we must be missing one of 12, 13, 14 and 15.

We can now do the same thing with the sums 3. 4, 5 and 6. The only way to make 3 is $1 + 2$, and the only way to make 4 is $1 + 3$. Then, this means 2 and 3 are both adjacent to 1, and therefore not adjacent to each other. The only ways to make 5 are $1 + 4$ and $2 + 3$, so the only possibility is $1 + 4$. This means 1 is adjacent to 2, 3 and 4, which are not adjacent to each other.

The only ways to make the sum of 6 are $1 + 5$ and $2 + 4$, but we already know all the neighbours of 1: 2, 3 and 4. Thus we cannot have $1 + 5$. Also, we know that 2 and 4 are on opposite corners of one face, so are not adjacent. This means that we cannot obtain the total 6 if we have totals of 3, 4 and 5.

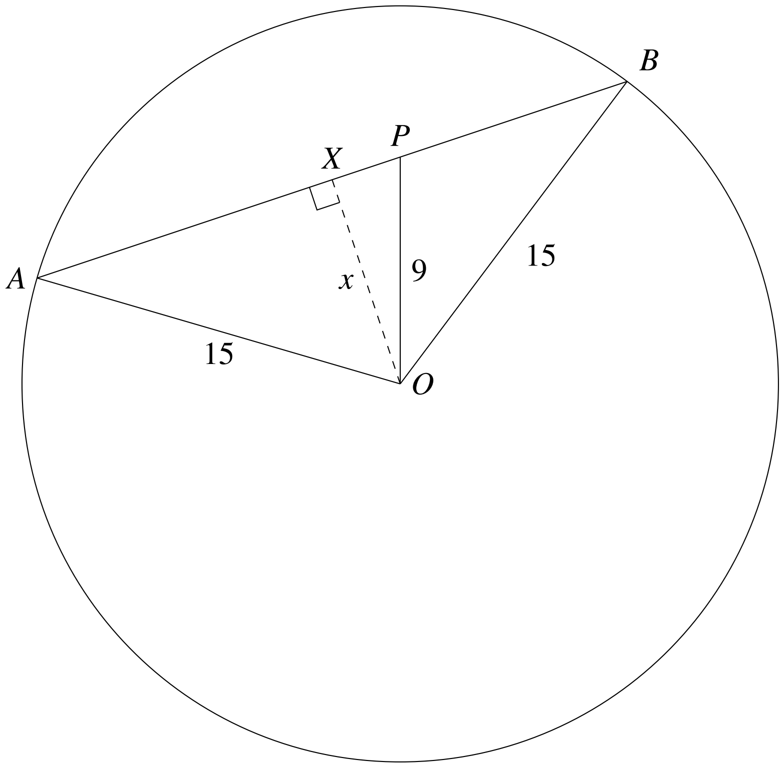
To sum up: of the thirteen possibilities, we are missing one of 3, 4, 5 and 6 and one of 12, 13, 14 and 15. This means at least two totals of the thirteen are missing, implying that there are at most eleven different totals. Hence there must be two totals that are the same and the answer to the question asked is: no, it is not possible.

> **4.** The point P is at distance 9 from the centre O of a circle of radius 15. What are the possible lengths of a chord of the circle passing through P?

Answer The length of a chord is between 24 and 30.

Solution

Let AB be a chord passing through P. Also let O be the centre of the circle and X the point on AB such that OX is perpendicular to AB. Let x be the length of OX.



Since X is the closest point on AB to O, we know that $0 \le OX \le OP$, or $0 \le x \le 9$.

As triangle ABO is isosceles with $AX = BX$, we can use Pythagoras' theorem to tell us that

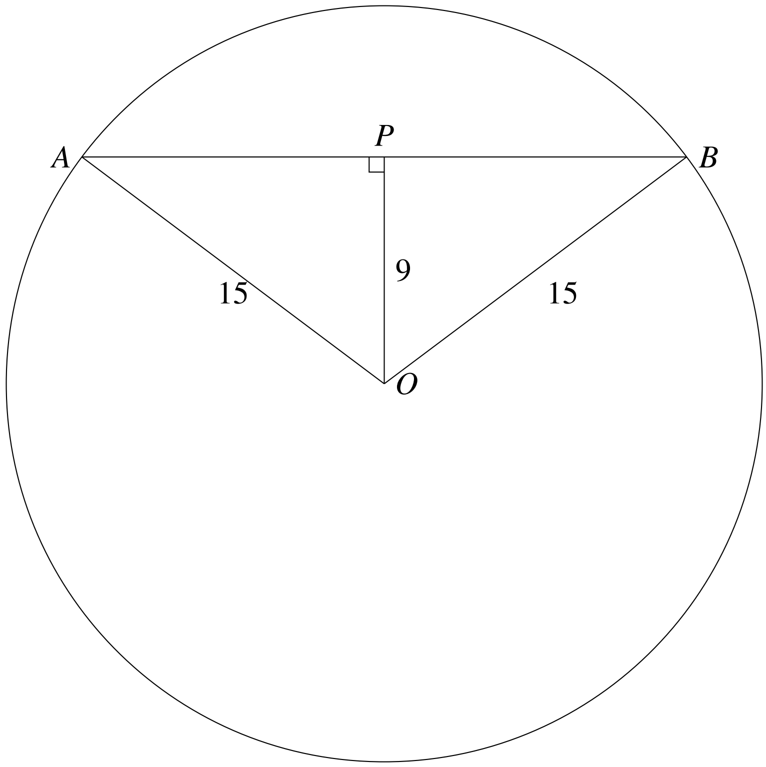
$$AB = 2\sqrt{OA^2 - OX^2} = 2\sqrt{15^2 - x^2}.$$

Then, since $0 \le x \le 9$, we have

$$2\sqrt{15^2 - 9^2} \le AB \le 2\sqrt{15^2 - 0^2}.$$

Doing the calculations, we get $24 \le AB \le 30$.

Now, we need to check that we can achieve all of these lengths. The value 30 is achievable by taking the diameter of the circle that passes through P.



In the case where AB is perpendicular to OP, Pythagoras' theorem tells us that $AP = \sqrt{OA^2 - OP^2} = \sqrt{15^2 - 9^2} = \sqrt{144} = 12$. Since AOB is isosceles, we know that $AP = BP$, so $AB = 24$.

This means we can achieve both $AB = 24$ and $AB = 30$. If we slide the chord around between these two positions, then the length will move from 24 to 30 and all the intervening lengths will be obtained.

5. For what non-negative integer values of m is $\sqrt{m + \sqrt{m + \sqrt{m + \ldots}}}$ also an integer?

Answer $m = y(y - 1)$ for y an integer.

Solution

Let us write

$$x = \sqrt{m + \sqrt{m + \sqrt{m + \ldots}}}$$

On squaring, we get

$$x^2 = m + \sqrt{m + \sqrt{m + \ldots}}$$

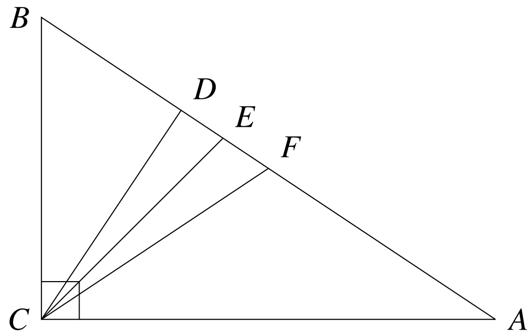
However, the right-hand term is exactly the same as the one we started with, so we can rewrite the last equation as

$$x^2 = m + x.$$

Then, $m = x^2 - x$.

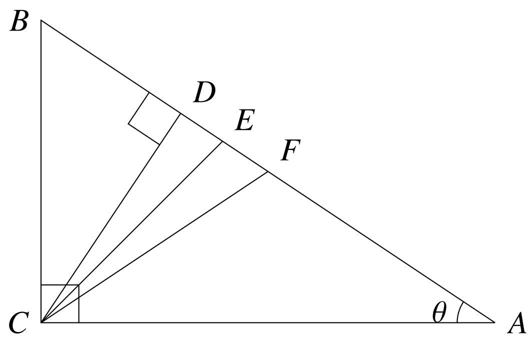
Consequently, for x to be an integer, m must be of the form $y^2 - y$ for some integer y. If m is of this form, then $x^2 - x = y^2 - y$, which rearranges to give $x^2 - y^2 - x + y = 0$. This can be factorised as $(x - y)(x + y - 1) = 0$, so $x = y$ or $x = 1 - y$. Whichever solution is correct, x is an integer, so any m of this form gives an integer solution.

6. In the right angled triangle ABC, CD is perpendicular to AB, CE is the angle bisector of $\angle ACB$ and F is the midpoint of the line AB. Prove that $\angle DCE = \angle ECF$.

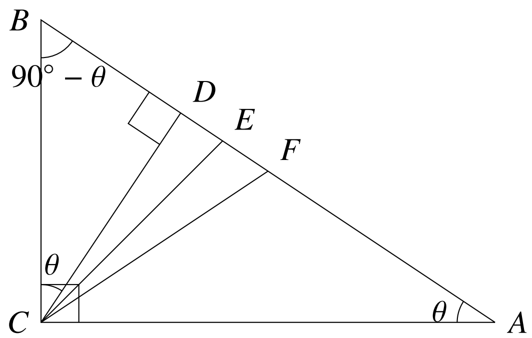


Solution

Let us label $\angle BAC = \theta$.

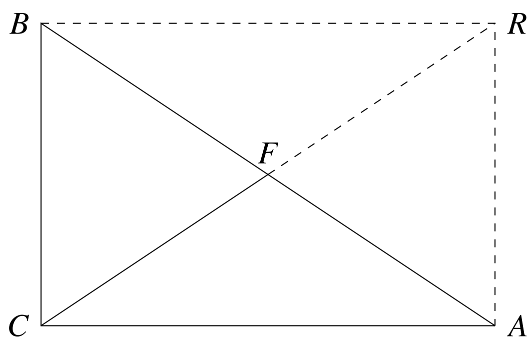


Since angles in triangle ABC sum to $180°$, we know that $\angle CBA = 90° - \theta$. Then, since angles in triangle BDC sum to $180°$ as well, $\angle BCD = \theta$ also.



Now, since CE is the angle bisector of $\angle BCA$, $\angle BCE = 45°$, so $\angle DCE = 45° - \theta$.

Next we want to consider the angle $\angle FCA$. Since F is the midpoint of AB, if we extend triangle ABC to a rectangle, then F is the midpoint of a diagonal. Therefore F is the point where the diagonals intersect.



But then we know that $FC = FA$, so FCA is an isosceles triangle. Consequently $\angle FCA = \angle FAC = \theta$. We also have $\angle ECA = 45°$, as CE is the angle bisector of $\angle BCA$. Then $\angle ECF = \angle ECA - \angle FCA = 45° - \theta$. Hence $\angle DCE = 45° - \theta = \angle ECF$, as required.

7. If $xy + x + y = 71$ and $x^2y + xy^2 = 880$, then what are the possible values of $x^2 + y^2$?

Answer 146 or 2993

Solution

> Initially it is not clear how to start this question. Thus it helps to play around with the equations you have been given to see what you can spot. In this case, you might spot that $x^2y + xy^2 = xy(x + y)$. This means that you have two equations in terms of xy and $x + y$.

Let us write p for xy and s for $x + y$. The original equations then become $p + s = 71$ and $ps = 880$.

Substituting the first of these into the second, we get $p(71 - p) = 880$, which rearranges to give

$$p^2 - 71p + 880 = 0.$$

Solving this quadratic, we get $(p - 55)(p - 16) = 0$, so $p = 55$ or $p = 16$. Then, $s = 71 - p$, so $s = 16$ or $s = 55$ respectively. This gives us two cases; in both we can check that the original equations must hold.

- **Case 1:** $p = 55$**,** $s = 16$

  As $p = xy$ and $s = x + y$, we know that $s^2 = x^2 + 2xy + y^2 = x^2 + y^2 + 2p$. This rearranges to give

  $$x^2 + y^2 = s^2 - 2p = 16^2 - 2 \times 55 = 146.$$

  All that remains is to check that we can find such an x and y. We know that:

  $$\begin{aligned} x + y &= 16; \\ xy &= 55. \end{aligned}$$

  Then, $x(16 - x) = 55$, which rearranges to give $x^2 - 16x + 55 = 0$. This factorises to give $(x - 5)(x - 11) = 0$. Hence either $x = 5$ (and $y = 11$) or $x = 11$ (and $y = 5$). It is easy to check that these indeed give the correct sum and product, so $x^2 + y^2 = 146$ is one possibility.

- **Case 2:** $p = 16$**,** $s = 55$

  We can similarly rearrange the equations to give

  $$x^2 + y^2 = s^2 - 2p = 55^2 - 2 \times 16 = 2993.$$

  Then we again need to check that there are suitable values of x and y. We get the equations:

  $$\begin{aligned} x + y &= 55; \\ xy &= 16. \end{aligned}$$

  So $x(55 - x) = 16$, which expands to $x^2 - 55x + 16 = 0$. We can solve this equation to get one of:

  $$x = \frac{55 + \sqrt{2961}}{2}, \qquad y = \frac{55 - \sqrt{2961}}{2};$$
  $$x = \frac{55 - \sqrt{2961}}{2}, \qquad y = \frac{55 + \sqrt{2961}}{2}.$$

In either case, you can check by calculation that the values of p and s are correct, so $x^2 + y^2 = 2993$ is the only other valid solution.

Hence, the possible values of $x^2 + y^2$ are 146 and 2993.

8. (a) Suppose that the integer x satisfies the equation

$$4x \equiv 6 \pmod{17}.$$

What are the possible values of x?

(b) Suppose that the positive integers r and n are coprime. Show that if

$$rx \equiv ry \pmod{n},$$

then $x \equiv y \pmod{n}$,

Hence show that there is an integer s such that $rs \equiv 1 \pmod{n}$.

The integer s is called the multiplicative inverse of r, sometimes written as r^{-1}. Multiplying by r^{-1} is similar to dividing by r.

(c) Do there exist any solutions to the equation $4x \equiv 6 \pmod{16}$?

(d) In what circumstances does there exist a solution for x in the equation

$$ax \equiv b \pmod{n}?$$

To answer this part of the question, give conditions satisfied by a, b and n.

Answer

(a) $x \equiv 10 \pmod{17}$
(c) No
(d) When the highest common factor of a and n is also a factor of b.

Solution

(a) We have $4x \equiv 6 \pmod{17}$ and we can multiply this congruence by 13. We know that $4x \times 13 \equiv 52x \equiv x \pmod{17}$ and $6 \times 13 \equiv 78 \equiv 10 \pmod{17}$. Therefore, $x \equiv 13 \times 4x \equiv 13 \times 6 \equiv 10 \pmod{17}$.

> It's not obvious why we should be able to find a number, in this case 13, which we can multiply by to get x on its own. Part (b) explains why there always is such a number. There are ways in which you can find this number, but they're well beyond the scope of this question. If you are interested, then you might want to look up Euclid's Algorithm and Extended Euclid's Algorithm.

(b) We know that $rx \equiv ry \pmod{n}$, so n divides $rx - ry = r(x - y)$. However, r and n have no common factors so this implies that n must divide $x - y$. But then $x \equiv y \pmod{n}$, as required.

Now, let us consider rt for each of $t = 0$, $t = 1$, ..., $t = n - 1$. As each t is different modulo n from any other, we know, from the first part, that the rt's must also be distinct. But then, as there are n integer values modulo n that are all different, one of them must be congruent to each remainder. Hence there is an s with $rs \equiv 1 \pmod{n}$.

(c) If $4x \equiv 6 \pmod{16}$, then $4x + 16k = 6$ for some integer k. But, considering this equation modulo 4, we get $0 \equiv 2 \pmod{4}$, which cannot happen. Thus there are no solutions.

(d) For any factor s that divides both a and n, we have that $b = ax - nk$ is a multiple of s. In particular, this tells us that the highest common factor of a and n must divide b.

If this is the case, then let us write $a = sA$, $n = sN$ and $b = sB$. Notice that, as s is the highest common factor of a and n, this tells us that A and N have no common factors.

Since $ax \equiv b \pmod{n}$, $sAx - sB$ is a multiple of sN. Equivalently, $Ax - B$ is a multiple of N, so $Ax \equiv B \pmod{N}$. Then part (b) tells us that there is an integer t such that $At \equiv 1 \pmod{N}$, so $x \equiv Atx \equiv Bt \pmod{N}$.

Now consider any such $x = Bt + kN$. We have $ax = AsBt + AskN = s(B \times At + Ak \times N)$. The right-hand term in the bracket is a multiple of N, and the left-hand term is $BAt \equiv B \times 1 \equiv B \pmod{N}$. Then multiplying by s tells us that $ax \equiv sB \equiv b \pmod{n}$, so we have a solution.

Therefore we get a solution exactly when s divides b, that is, when the highest common factor of a and n is a factor of b.

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Julia Robinson

Sheet 5

Solutions and comments

This programme of the Mentoring Scheme is named after Julia Robinson (1919–1985).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Robinson_Julia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

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1. An engineer can finish building a road in three days with her present supply of machines. With three more machines, she could finish in just two days. If all her machines work at the same rate, then how many days would it take her to finish the road with only one machine?

Answer 18 days

Solution

Let us suppose that the engineer currently has m machines, and that it would take r days to make the road with just one machine. Then, with her current machines, the engineer would take $\frac{r}{m}$ days, so we know that

$$\frac{r}{m} = 3.$$

With $m + 3$ machines, it takes 2 days, so

$$\frac{r}{m+3} = 2.$$

Multiplying up both of these equations, we get:

$$\begin{aligned} r &= 3m; \\ r &= 2m + 6. \end{aligned}$$

This tells us that $3m = 2m + 6$, so, subtracting $2m$ from each side, we get $m = 6$.

Now, $r = 3m = 3 \times 6 = 18$, so the task would take 18 days with a single machine.

2. How many factors does the number of seconds in a week have?

Answer 192

Solution

There are 60 seconds in a minute, 60 minutes in an hour, 24 hours in a day and 7 days in a week. Therefore there are $60 \times 60 \times 24 \times 7$ seconds in a week.

Now, if we want to work out the prime factorisation of this, we can work out the prime factors of each of the factors. We know that:

$$\begin{aligned} 60 &= 2^2 \times 3 \times 5; \\ 60 &= 2^2 \times 3 \times 5; \\ 24 &= 2^3 \times 3; \\ 7 &= 7. \end{aligned}$$

This means that the prime factorisation of the number of seconds in a week is

$$\begin{aligned} 60 \times 60 \times 24 \times 7 &= \left(2^2 \times 3 \times 5\right) \times \left(2^2 \times 3 \times 5\right) \times \left(2^3 \times 3\right) \times 7 \\ &= 2^7 \times 3^3 \times 5^2 \times 7. \end{aligned}$$

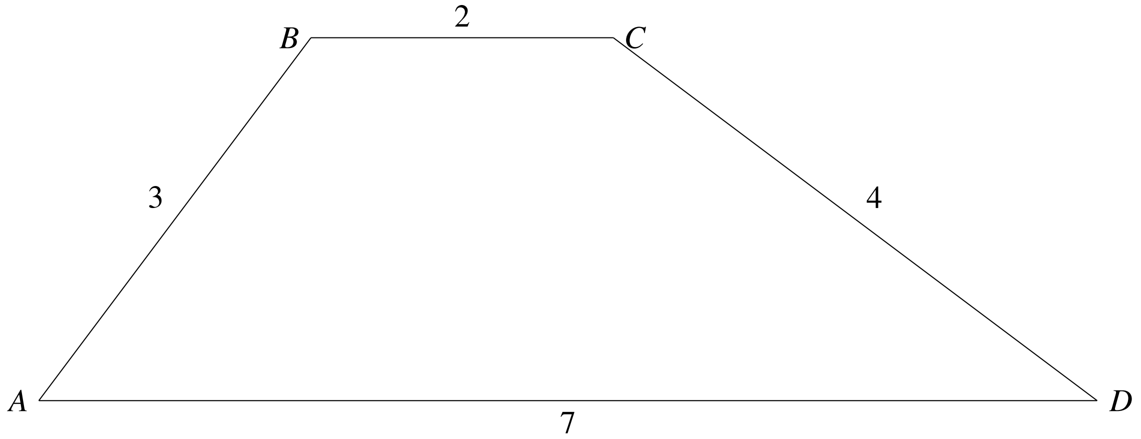
Any factor of this number must therefore be of the form $2^a \times 3^b \times 5^c \times 7^d$, where $0 \le a \le 7$, $0 \le b \le 3$, $0 \le c \le 2$ and $0 \le d \le 1$. Any such choice of a, b, c and d is a factor, since $2^a 3^b 5^c 7^d \times 2^{7-a} 3^{3-b} 5^{2-c} 7^{1-d} = 2^7 3^3 5^2 7$. Moreover, as these choices have different prime factorisations, these are all different numbers.

There are 8 choices of a, 4 choices of b, 3 choices of c and 2 choices of d. This means that there are $8 \times 4 \times 3 \times 2 = 192$ different factors.

> **3.** A trapezium $ABCD$ has sides BC and AD parallel. If $AB = 3$, $BC = 2$, $CD = 4$ and $DA = 7$, then what is the area of the trapezium?

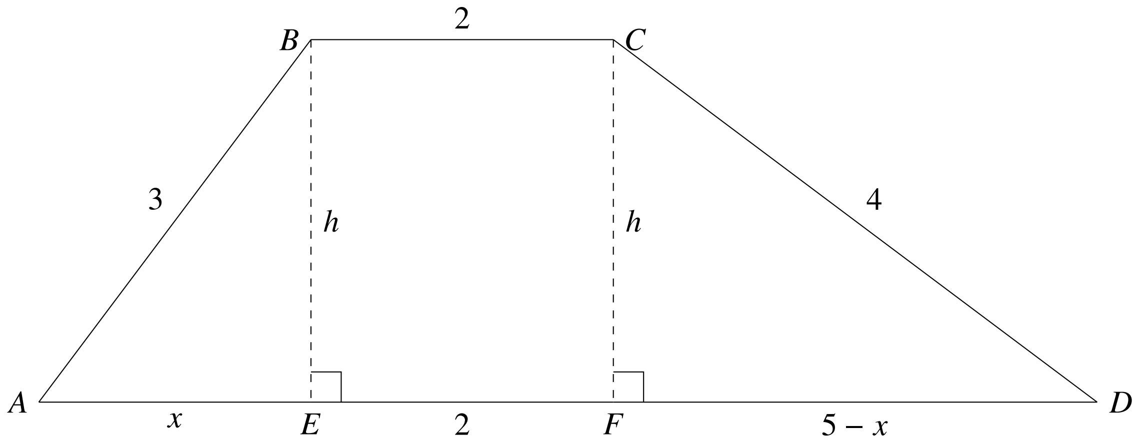
Answer 10.8

Solution



We need to work out the height of the trapezium and there are two main ways of doing this.

Working algebraically, we can label the height as h and also draw in the perpendiculars from B and C onto AD.



Now, we can apply Pythagoras' theorem in triangles ABE and CFD to give the following equations:

$$\begin{aligned} x^2 + h^2 &= 3^2; \\ (5 - x)^2 + h^2 &= 4^2. \end{aligned}$$

Subtracting the first equation from the second, we get

$$(5 - x)^2 - x^2 = 4^2 - 3^2.$$

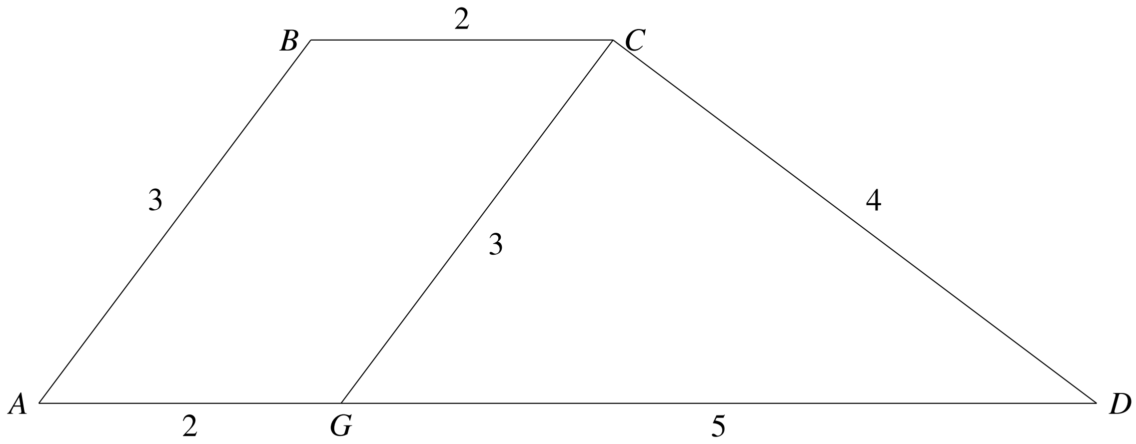
Expanding the bracket and simplifying, we get

$$10x = 18.$$

Therefore $x = 1.8$, so $h = \sqrt{3^2 - 1.8^2} = \sqrt{9 - 3.24} = \sqrt{5.76} = 2.4$. Recall that, if the parallel sides of a trapezium have sides a and b and the perpendicular height is h, then the area of the trapezium is $\frac{1}{2}(a + b)h$. Therefore the area of our trapezium is $\frac{1}{2} \times (7 + 2) \times 2.4 = 10.8$

Alternative

Draw a line through C parallel to AB, and let G be the point where it meets AD. Then, $ABCG$ is a parallelogram, so $AG = BC = 2$ and $EC = AB = 3$. Then $GD = AD - AG = 7 - 2 = 5$.



Now, we can consider the triangle CGD. Since we know that $3^2 + 4^2 = 5^2$, $CG^2 + CD^2 = GD^2$ and (the converse of) Pythagoras' theorem tells us that $\angle GCD$ is a right angle. It follows that the area of CGD is $\frac{1}{2} \times CD \times CG = \frac{1}{2} \times 3 \times 4 = 6$.

Next, if h is the perpendicular height of C above GD, we know that the area of CGD is also equal to $\frac{1}{2}h \times GD$, so $6 = \frac{5}{2}h$, and $h = 2.4$. Then the area of the parallelogram is $AG \times h = 2 \times 2.4 = 4.8$.

The total area is then the sum of the areas of the triangle and the parallelogram, so is $6 + 4.8 = 10.8$.

4. *All the variables involved in this question are assumed to be positive integers.*

(a) Suppose that r divides both a and b. Show that r divides both $a + b$ and $a - b$.

(b) If s is the highest common factor of a and b, then show that s is also the highest common factor of a and $b - a$.

(c) What is the highest common factor of 30073 and 83143?

Answer

(c) 1769

Solution

(a) As r divides a, we can write $a = rx$. Similarly, we can write $b = ry$. Then $a - b = rx - ry = r(x - y)$ is divisible by r. Similarly, $a + b = rx + ry = r(x + y)$ is also divisible by r.

(b) Part (a) tells us that, as s is a factor of both a and b, it is also a factor of $b - a$. It is therefore a common factor of a and $b - a$.

If t divides both a and $b - a$, then part (a) tells us that t must also divide $a + (b - a) = b$, so t is also a common factor of a and b. As s is the highest common factor of a and b, this tells us that $t \leq s$. Therefore s is the largest common factor of a and $b - a$.

(c) The highest common factor of 30073 and 83143 is also the highest common factor of 30073 and $83143 - 30073 = 53070$, by applying part (a). If we continue to apply this result, we obtain the

following chain.

$$\begin{aligned} HCF(30073, 83143) &= HCF(30073, 53070) \\ &= HCF(30073, 53070 - 30073) = HCF(30073, 22997) \\ &= HCF(30073 - 22997, 22997) = HCF(7076, 22997) \\ &= HCF(7076, 22997 - 7076) = HCF(7076, 15921) \\ &= HCF(7076, 15921 - 7076) = HCF(7076, 8845) \\ &= HCF(7076, 8845 - 7076) = HCF(7076, 1769). \end{aligned}$$

By continuing to use the same procedure, we have

$$HCF(7076, 1769) = HCF(7076 - 4 \times 1769, 1769) = HCF(0, 1769) = 1769.$$

Therefore the highest common factor of the original pair of integers is 1769.

> This process is known as *Euclid's Algorithm*, and it is a quick method of calculating highest common factors without having to work out the prime factors of the number. It is usually applied (as in part (c)) by subtracting multiples of one number from the other. You can find out more about it online.

5. (a) What are the possible remainders when a square number is divided by 8? What are the possible remainders when the divisor is 24?

(b) Show that all prime numbers apart from 2 and 3 occur as terms in the sequence $a_n = \sqrt{24n+1}$.

Answer

(a) 0, 1 and 4
0, 1, 4, 9, 12 and 16

Solution

(a) Let us work modulo 8, writing down the squares of the possible values as follows.

$x \pmod 8$	$x^2 \pmod 8$
0	0
1	1
2	4
3	$9 \equiv 1$
4	$16 \equiv 0$
5	$25 \equiv 1$
6	$36 \equiv 4$
7	$49 \equiv 1$

Therefore the possible remainders are 0, 1 and 4.

We can perform the same operation when we are working modulo 24. We can also speed things up by noticing that $(-x)^2 = x^2$, so $(24 - x)^2 \equiv (-x)^2 = x^2$. The new table is then as follows.

$x \pmod{24}$	$x^2 \pmod{24}$
0	0
1, 23	1
2, 22	4
3, 21	9
4, 20	16
5, 19	$25 \equiv 1$
6, 18	$36 \equiv 12$
7, 17	$49 \equiv 1$
8, 16	$64 \equiv 16$
9, 15	$81 \equiv 9$
10, 14	$100 \equiv 4$
11, 13	$121 \equiv 1$
12	$144 \equiv 0$

Therefore the possible remainders are 0, 1, 4, 9, 12 and 16.

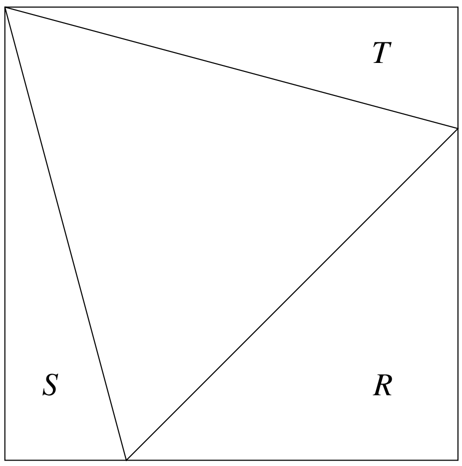
(b) First, notice that an integer x is a term in the sequence exactly if $x = \sqrt{24n+1}$ for some positive integer n. This is exactly the same as saying that $x^2 = 24n + 1$, or $x^2 \equiv 1 \pmod{24}$.

Suppose p is a prime number and is neither 2 nor 3. It then cannot be divisible by 2 or 3. Since $24 = 2^3 \times 3$, $24 + a$ is a multiple of 2 precisely when a is a multiple of 2. Similarly, $24 + a$ is a multiple of 3 if and only if a is a multiple of 3.

This means that p cannot be congruent to 0, 2, 4,6, 8, 10, 12, 14, 16, 18, 20 or 22, as these are even. Likewise, p cannot be congruent to 3, 9, 15 or 21, as these are multiples of 3. This leaves 1, 5, 7, 11, 13, 17, 19 and 23.

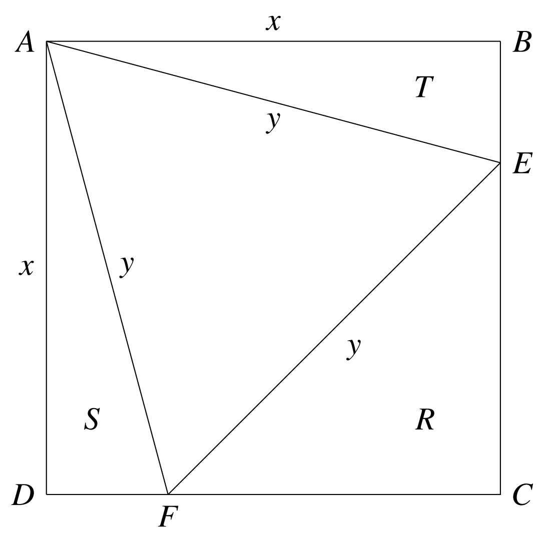
However the (second) table in part (a) tells us that all these remainders square to give $1 \pmod{24}$, so we know that $p^2 \equiv 1 \pmod{24}$. Therefore p must be in the given sequence.

6. A square has an equilateral triangle drawn in it as shown. Show that Area R = Area S + Area T.



Solution

Let us start by labelling some of the points (see the diagram). Also denote the side length of the square by x and that of the triangle by y.



Now, since $\angle ADC$ is a right angle, we can apply Pythagoras' theorem to triangle ADF to tell us that $FD = \sqrt{AF^2 - AD^2} = \sqrt{y^2 - x^2}$. We can repeat this in triangle ABE to tell us that $EB = \sqrt{y^2 - x^2}$.

Then, since $ABCD$ has side length x, we have $FC = x - \sqrt{y^2 - x^2}$ and likewise $EC = x - \sqrt{y^2 - x^2}$.

Now consider triangle ECF. Since this is right angled, we can again apply Pythagoras' theorem to obtain $EF^2 = FC^2 + EC^2$. This is

$$y^2 = 2\left(x - \sqrt{y^2 - x^2}\right)^2.$$

Expanding the right-hand side, we get

$$y^2 = 2x^2 - 4x\sqrt{y^2 - x^2} + 2y^2 - 2x^2.$$

Collecting like terms and simplifying, we have

$$y^2 = 4x\sqrt{y^2 - x^2} \tag{1}$$

In triangle ECF, since $FC = EC$, $2FC^2 = y^2$, so $FC = \frac{1}{\sqrt{2}}y$

We are now in a position to calculate the areas of the labelled regions.

$$\begin{aligned}
R &= \frac{1}{2} \times FC \times EC = \frac{1}{2} \times \frac{y}{\sqrt{2}} \times \frac{y}{\sqrt{2}} = \frac{1}{4}y^2 \\
S &= \frac{1}{2} \times AD \times FD = \frac{1}{2} \times x \times \sqrt{y^2 - x^2} = \frac{1}{2}x\sqrt{y^2 - x^2} \\
T &= \frac{1}{2} \times AB \times EB = \frac{1}{2} \times x \times \sqrt{y^2 - x^2} = \frac{1}{2}x\sqrt{y^2 - x^2}
\end{aligned}$$

Now use 1, to get

$$S + T = x\sqrt{y^2 - x^2} = \frac{1}{4}y^2 = R.$$

7. Find a pair (a, b) of positive integers such that $\left(\sqrt[3]{a} + \sqrt[3]{b} - 1\right)^2 = 49 + 20\sqrt[3]{6}$.

ANSWER (48, 288)

Solution

Expanding the given equation yields

$$\sqrt[3]{a^2} + \sqrt[3]{b^2} + 1 + 2\sqrt[3]{ab} - 2\sqrt[3]{a} - 2\sqrt[3]{b} = 49 + 20\sqrt[3]{6}.$$

We need to have a term involving $\sqrt[3]{6}$ on the left-hand side of this equation. At present there are several cube roots on this left-hand side, suggesting that we should try to make one of them of the form (a cube $\times \sqrt[3]{6}$). Let us try $a = 6r^3$. This produces a term $\sqrt[3]{a^2} = r^2\sqrt[3]{36}$, so it makes sense to construct a term to deal with the $\sqrt[3]{36}$. A possibility is to set $b = 36s^3$.

Now we have $\sqrt[3]{a} = \sqrt[3]{6r^3} = r\sqrt[3]{6}$ and $\sqrt[3]{b} = \sqrt[3]{36s^3} = s\sqrt[3]{36}$ and the original equation becomes

$$\left(r\sqrt[3]{6} + s\sqrt[3]{36} - 1\right)^2 = 49 + 20\sqrt[3]{6}.$$

Expanding this, and remembering that $\sqrt[3]{216} = 6$, we get

$$r^2\sqrt[3]{36} + 6s^2\sqrt[3]{6} + 12rs - 2r\sqrt[3]{6} - 2s\sqrt[3]{36} + 1 = 49 + 20\sqrt[3]{6}.$$

Then we can equate the coefficients of the terms $\sqrt[3]{6}$ and $\sqrt[3]{36}$ to obtain:

$$\begin{aligned} 12rs + 1 &= 49; \\ 6s^2 - 2r &= 20; \\ r^2 - 2s &= 0. \end{aligned}$$

The final equation implies that $2s = r^2$, which can be substituted into the first equation. This becomes $6r^3 + 1 = 49$, meaning $r^3 = 8$ or $r = 2$. Then, $s = \frac{1}{2}r^2 = 2$, so we have a possible solution. For this to be a valid solution, we need the middle equation to be correct. Making the substitutions yields

$$6s^2 - 2r = 6 \times 2^2 - 2 \times 2 = 24 - 4 = 20.$$

Thus the middle equation is satisfied and we have a solution. Since $a = 6r^3$ and $b = 36s^3$, the required values are $a = 6r^3 = 6 \times 2^3 = 48$ and $b = 36s^3 = 36 \times 2^3 = 288$. Since the initial steps of the solution involved a certain amount of guesswork, it would be prudent to go back to the original equation and to check that is is indeed satisfied.

8. *This question follows on from Question 8 on last month's sheet.*

(a) Suppose that x satisfies the following equations:

$$\begin{aligned} x &\equiv 3 \pmod{5}; \\ x &\equiv 4 \pmod{8}. \end{aligned}$$

What are the possible values of x?

(b) Suppose that m and n are coprime and that x satisfies the following equations;

$$\begin{aligned} x &\equiv a \pmod{m}; \\ x &\equiv b \pmod{n}. \end{aligned}$$

Show that these equations have a solution regardless of the values of the integers a and b.

Must this solution be unique modulo mn?

This is part of what is known as the Chinese Remainder Theorem.

ANSWER

(a) $x \equiv 28 \pmod{40}$

SOLUTION

(a) Since $x \equiv 4 \pmod 8$, we can write $x = 8n + 4$ for some integer n. Then, $8n + 4 \equiv 3 \pmod 5$ from the second equation, so $3n \equiv 4 \pmod 5$.

We can multiply both sides of this congruence by 2 (since 2 is the inverse of 3 in arithmetic modulo 5), to get $6n \equiv 8 \pmod 5$. Simplifying tells us that $n \equiv 3 \pmod 5$, so we can write $n = 5r + 3$ for some integer r.

Then, $x = 8n + 4 = 8(5r + 3) + 4 = 40r + 28$, so $x \equiv 28 \pmod{40}$. But, if x is of this form, then $x = 28 + 40r$, which means that x satisfies both the equations. Therefore x satisfies both equations exactly when $x \equiv 28 \pmod{40}$.

(b) Since $x \equiv a \pmod m$, we can write $x = rm + a$ for some integer r. Substituting into the second equation, $rm + a \equiv b \pmod n$. Now, let v be the inverse of m modulo n, which we know exists from Question 8 on the last sheet. This tells us that $vrm + va \equiv vb \pmod n$, or $r \equiv vb - va \pmod n$.

Therefore r is of the form $sn + vb - va$ for some integer s. It follows that

$$x = (sn + vb - va)m + a = smn + (vbm - vam + a)$$

and therefore the single possible solution modulo mn is $x \equiv vbm - vam + a \pmod{mn}$.

Notice that, if we reduce this modulo m, we get $x = smn + (vbm - vam + a) \equiv a \pmod n$. If we reduce modulo n, then we get $x = smn + vbm - vam + a \equiv (b - a)vm + a \equiv b - a + a \equiv b \pmod n$, since $mv \equiv 1 \pmod n$ by the definition of v.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Julia Robinson

Sheet 6

Solutions and comments

This programme of the Mentoring Scheme is named after Julia Robinson (1919–1985).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Robinson_Julia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

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1. Arrange the numbers 1 to 9 in the grid below, such that each row and column has the product shown.

	6	192	315
20			
126			
144			

Answer

	6	192	315
20	1	4	5
126	3	6	7
144	2	8	9

Solution

Only the top row has a factor of 5 and similarly only the right hand column has a factor of 5, so 5 must go in the top right hand box. Likewise, only the middle row and the right hand column have factors of 7, so 7 must go in the middle right box. For 8, only the middle column and the bottom rows have it as a factor, so 8 must go in the middle box of the bottom row.

	6	192	315
20			5
126			7
144		8	

Then 35 times the bottom right box must give 315, so this box must contain 9. Again, the bottom left box multiplied by 72 gives 144, so this box must contain 2.

	6	192	315
20			5
126			7
144	2	8	9

Now, the number in the top left box must be a common factor of both $6 \div 2 = 3$ from the first column, and $20 \div 5 = 4$ from the first row. The only common factor of 4 and 3 is 1, so this must go in the top left corner.

	6	192	315
20	1		5
126			7
144	2	8	9

From the top row, the top middle box must be $20 \div (1 \times 5) = 4$. From the left hand column, the middle left box must be $6 \div (1 \times 2) = 3$. Then, from the middle row, the centre box must be $126 \div (3 \times 7) = 6$. Therefore the only possible table is as follows.

	6	192	315
20	1	4	5
126	3	6	7
144	2	8	9

By multiplying out each of the rows and columns, you can see that it is indeed valid.

2. Each face of a cube is going to be painted with exactly one of a number of colours in such a way that no two adjacent faces are painted the same colour.

How many ways are there to paint the cube if there are:

(a) 3 colours available;

(b) 4 colours available.

Two colourings are the same if the cubes produced can be rotated to look the same.

Answer

(a) 1

(b) 10

Solution

(a) Let us call the three colours red, green and blue. At each vertex of the cube, three faces all meet, so they must be one of each colour. Consider the face opposite the red face: this must also be red as it touches the blue and green faces. The same is true of the faces opposite the blue and green faces, they must be blue and green respectively.

This means there is only one possible way of painting the cube, with each opposite pair of faces the same colour.

(b) Suppose the fourth colour is yellow. We can divide the painted cubes into three cases, depending on the number of faces that are painted yellow.

i. If there are no yellow faces, we are in the same situation as in part (a), so there is only one way to paint the cube.

ii. If there is exactly one yellow face, then there are three choices for the face opposite it. If the opposite face was, say, green, then none of the other four faces could be green or yellow, as they both touch these faces.

In order to colour the other faces using the remaining two colours, this can only be achieved by making the opposite faces the same colour. Then, the two possible orientations rotate onto each other.

Therefore there are 3 possible cubes with one yellow face, enumerated by the colour of the face opposite to yellow.

iii. If there are two yellow faces, these cannot be adjacent, so they must be opposite each other. There are then four remaining faces, and three possible colours for them.

If only two colours are used, these must be opposite each other, as in part i. There are three possible choices for the missing colour, so this gives three more cubes.

Otherwise, all the colours are used. Exactly one of these must occur twice, as there are four faces and three colours. There are three ways of choosing this colour. Then the other two faces must be the other two colours, and the two colourings can be rotated by $180°$ onto each other.

Therefore, there are $1 + 3 + 3 + 3 = 10$ possible colourings.

3. Suppose n is a positive integer. If $3n + 1$ divides $27n + 37$, then what are the possible values of n?

Answer $n = 1$ or $n = 2$ or $n = 9$

Solution

If $3n + 1$ divides $27n + 37$, then we can use the result from Question 4 on the previous sheet to see that $3n + 1$ must also divide $(27n + 37) - 9(3n + 1) = 27n + 37 - 27n - 9 = 28$.

The factors of 28 are 1, 2, 4, 7, 14, 28, −1, −2, −4, −7, −14 and −28. Since n is a positive integer, $n \geq 1$, so $3n + 1 \geq 4$. Since n is an integer, $3n + 1$ must leave a remainder of 1, when divided by 3. This means the possible values of $3n + 1$ are 4, 7 and 28.

These correspond to $n = 1$, $n = 2$ and $n = 9$. We can check that these are solutions:

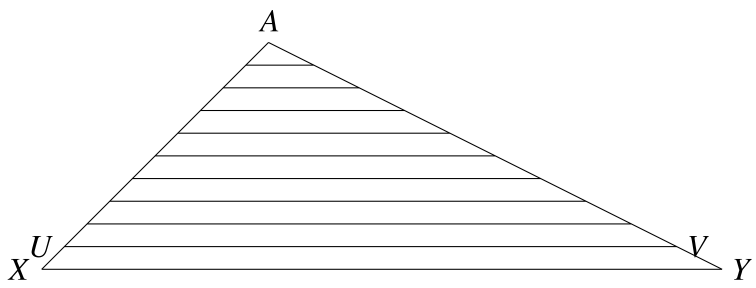
- If $n = 1$, then $3n + 1 = 4$ and $27n + 37 = 64 = 4 \times 16$.
- If $n = 2$, then $3n + 1 = 7$ and $27n + 37 = 91 = 7 \times 13$.
- If $n = 9$, then $3n + 1 = 28$ and $27n + 37 = 280 = 28 \times 10$.

Hence, these are all valid solutions.

4. Nine lines are drawn parallel to the base of a triangle to divide each of the other sides into 10 equal segments. If the area of the largest region is 38, find the area of the original triangle.

Answer 200

Solution



Since all the lines drawn are parallel, all the triangles with one of the parallel lines as the base and A as the vertex are similar.

Therefore, the areas of the triangles AUV and AXY are in the ratio of the squares of the sides.

Using square brackets to denote the area of a shape, we may express this statement as $[AUV] : [AXY] = AU^2 : AX^2$. However, we also know that the area of $UVYX$ is 38, and $[AUV] + [UVYX] = [AXY]$. Also, as the lines are equally spaced, $AU : AX = 9 : 10$. Therefore:

$$\frac{[AUV]}{[AUV] + 38} = \left(\frac{9}{10}\right)^2$$

Rearranging, we get:

$$100[AUV] = 81[AUV] + 81 \times 38$$

Therefore, $19[AUV] = 81 \times 38$, so $[AUV] = 81 \times 2 = 162$. Hence, the area of AXY is $162 + 38 = 200$.

5. Find all real numbers x such that $\sqrt{4x - 3} - \sqrt{13 - 4x} = 2$.

Answer $x = 3$

Solution

$$\sqrt{4x-3}-\sqrt{13-4x}=2$$

Squaring both sides produces

$$4x-3+13-4x-2\sqrt{(4x-3)(13-4x)}=4.$$

On simplifying we have

$$6=2\sqrt{(4x-3)(13-4x)}.$$

Dividing by 2 and squaring yields

$$9=(4x-3)(13x-4).$$

On expanding the brackets and collecting like terms, we have

$$16x^2-64x+48=0.$$

We can now divide by 16 to obtain

$$x^2-4x+3=0.$$

Finally, this simple quadratic can be factorised as

$$(x-3)(x-1)=0.$$

Hence the only possible solutions are $x = 1$ and $x = 3$. However, we do not know that these are actual solutions, only that there are no others. Thus we need to check to see if they work.

If $x = 1$, then $\sqrt{4x-3}-\sqrt{13-4x}=\sqrt{4-3}-\sqrt{13-4}=\sqrt{1}-\sqrt{9}=1-3=-2$, so this isn't a solution.

If $x = 3$, then $\sqrt{4x-3}-\sqrt{13-4x}=\sqrt{12-3}-\sqrt{13-12}=\sqrt{9}-\sqrt{1}=3-1=2$, so this is a valid solution.

Therefore the only solution is $x = 3$.

6. Let $L_1, L_2, \ldots, L_{100}$ be a collection of lines in a two-dimensional plane.

(a) What is the maximum number of points of intersection of these lines that we can obtain?

(b) What is the maximum number of points of intersection if all the lines of the form L_{4k} for k a positive integer are parallel, and all the lines of the form L_{4n-3} for n a positive integer meet at a point?

Answer

(a) 4950

(b) 4351

Solution

(a) Since each of the L_i are straight lines, each pair can intersect at most once. How many of these pairs are there? There are 100 choices for the first line, and 99 for the second line. However, we can pick these in either order, so there are $100 \times 99 \div 2 = 4950$ pairs of lines.

Can we place the lines to obtain this maximum number of points of intersections? Since two lines will intersect as long as they are not parallel, we can place the lines as long as each line is not parallel to any of the others that are already placed, and does not pass through any of the current points of intersection.

There are infinitely many directions we can choose for a line, and only finitely many already placed, so we can pick a direction different from all of those. There are then infinitely many different lines in this direction, and only finitely many points of intersection to avoid, so we can definitely arrange for all the intersections to be different.

Therefore the maximum number of intersections is 4950.

(b) For the lines L_{4k}, these are parallel, so no pair of these is intersecting. There are $25 \times 24 \div 2 = 300$ pairs of such lines. Likewise there are 300 pairs of lines each of which are of the form L_{4n-3}, and these all intersect in a single point. This means that we must lose at least $300 + 300 - 1 = 599$ points of intersection. This would leave $4950 - 599 = 4351$ points.

Can we achieve this? Place down all the lines of the form L_{4n-3} passing through the same point. Then, pick a direction different to the directions of all of these lines and place 25 lines in this direction that do not pass through the point of intersection. Placing all of the remaining lines as in the first part of the question then achieves this maximum number of intersections.

7. Suppose that m and n are positive integers. If $\frac{m^2+n^2}{mn}$ is also an integer, what possible values can this integer take?

Answer Only 2

Solution

Let x be the highest common factor of m and n. Then write $m = rx$ and $n = sx$, and note that r and s are coprime integers. Then, it must be the case that

$$\frac{m^2+n^2}{mn} = \frac{r^2x^2+s^2x^2}{rsx^2} = \frac{r^2+s^2}{rs}$$

is still an integer.

Consequently $r^2 + s^2$ has a factor of r. Since r^2 clearly has a factor of r, then so does s^2. But we also know that r and s are coprime, so r and s^2 are also coprime. For r and s^2 to be coprime but also for r to divide s^2 requires that $r = 1$.

But then, $\frac{1+s^2}{s} = s + \frac{1}{s}$ is an integer, so $\frac{1}{s}$ is an integer, so $s = 1$ also. This means that $m = n = x$. Then

$$\frac{m^2+n^2}{mn} = \frac{x^2+x^2}{x^2} = 2.$$

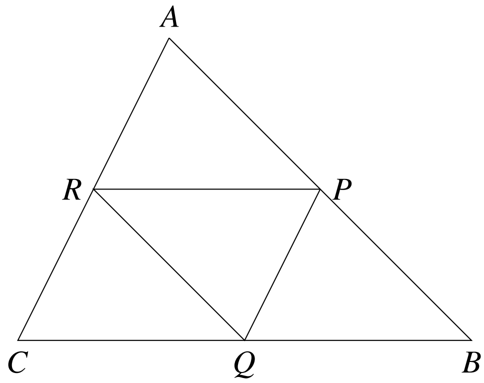
Therefore this is the only possible solution.

8. (a) Suppose that ABC is a triangle. Let P be the midpoint of AB, Q the midpoint of AC and R the midpoint of BC. Prove that all the triangles ARP, BPQ, CQR and RQP are congruent.

(b) Suppose that ABC is a triangle. Let D be the centre of the square constructed on AB, and E the centre of the square constructed on AC. If F is the centre of BC, prove that triangle DFE is isosceles and right angled.

Solution

(a) Since P, Q, R are all midpoints, we know that $AP = PB$, $BQ = QC$ and $CR = RA$.

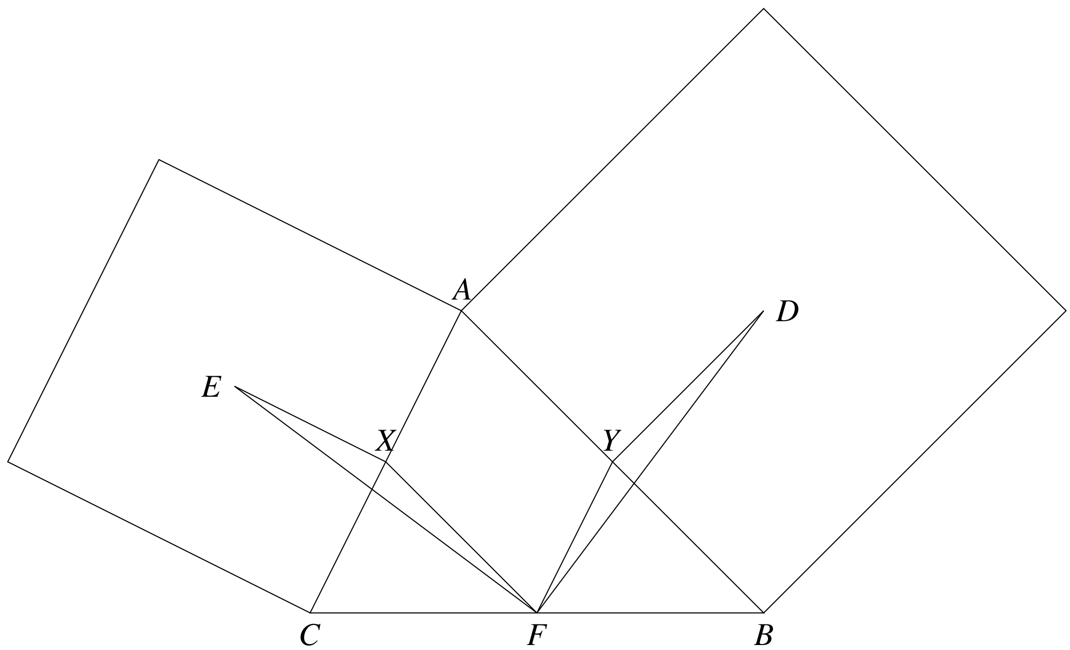


In particular, we can write this as $CR = \frac{1}{2}CA$ and $CQ = \frac{1}{2}CB$. Also $\angle RCQ = \angle ACB$ since they are the same angle. This tells us that triangle $CQR = RCQ$ is an enlargement of triangle ACB with scale factor $\frac{1}{2}$.

We can repeat this procedure on the triangles ARP and BPQ to find that both these triangles are enlargements of ABC with scale factor $\frac{1}{2}$. Hence all three triangles are congruent.

Since RCQ is an enlargement of ABC by scale factor $\frac{1}{2}$, we know that $RQ = \frac{1}{2}AB$. In the same way, $RP = \frac{1}{2}CB$ and $PQ = \frac{1}{2}AC$. This means each side of triangle PQR is half of the corresponding side of ABC. Hence PQR is also an enlargement of ABC with scale factor $\frac{1}{2}$. Therefore, by "side-side-side", all four triangles are congruent.

(b) Let X and Y be the midpoints of AC and AB respectively. Then we can draw in the lines XE, XF, YD and YF, to produce the diagram below.



Now since E is the centre of the square constructed on AC, $EX = XC$. Since, from part (a), XCF is congruent to YFB, we know that $XC = YF$. Hence $EX = YF$.

Likewise, as D is the centre of the square constructed on AB, $DY = YB$. From the congruent triangles, $XF = YB = DY$.

Since X is the midpoint of AC and E is the centre of the square, $\angle EXC = 90°$. Likewise $\angle DYB = 90°$. As triangles CXF and FYB are congruent, $\angle CXF = \angle FYB$. Hence $\angle EXF = \angle EXC + \angle CXF = 90° + \angle FYB = \angle FYB + \angle BYD = \angle FYD$.

This means that $EX = FY$, $\angle EXY = \angle FYD$ and $XF = YD$. Thus, by "side-angle-side", triangles EXF and FYD are congruent. In particular, this tells us that $EF = FD$.

As EXF and FYD are congruent, $\angle YFD = \angle XEF$. Likewise, from part (a), XFY is congruent to FXC, so $\angle XFY = \angle FXC$.

Then, $\angle EFD = \angle EFX + \angle XFY + \angle YFD = \angle EFX + \angle FXC + \angle XEF$. Then, as the angles in triangle EFX add to $180°$, $\angle EFX + \angle FXC + \angle EXC + \angle XEF = 180°$. This means $\angle EFD = 180° - \angle EXC = 180° - 90° = 90°$.

Therefore DEF is a right-angled isosceles triangle.

United Kingdom Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Julia Robinson

Sheet 7

Solutions and comments

This programme of the Mentoring Scheme is named after Julia Robinson (1919–1985).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Robinson_Julia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

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1. The integer m is the first of five consecutive integers that are multiples of 3, 4, 5, 6 and 7. If $m > 7$, then what is the smallest possible value of m?

ANSWER $m = 423$

SOLUTION

We know that m is a multiple of 3, $m + 1$ is a multiple of 4, $m + 2$ is a multiple of 5, $m + 3$ is a multiple of 6 and $m + 4$ is a multiple of 7.

For each of these, this is the same as saying that $m - 3$ is a multiple of 3, 4, 5, 6 and 7. This means that $m - 3$ is a multiple of the *least common multiple* of 3, 4, 5, 6 and 7. The LCM is 420. Since $3 < 7$, the next smallest possible value of m is 423.

2. A set of marbles is arranged in a pile. The base of the pile is a rectangle containing 1080 marbles. Each marble in the higher layers touches four marbles of the layer below, and contains as many marbles as possible. The top layer of the pile is a rectangle with 400 marbles. If the fifth layer up is a rectangle whose length is four times its width, how many marbles are in the middle layer of the pile?

ANSWER 715

SOLUTION

Suppose that the bottom rectangle is x marbles wide and y marbles deep. Since there are 1080 marbles in this layer, we know that $xy = 1080$.

Each time you go up a layer, there is one marble less in each direction. This means the fifth layer up has width $x - 4$ and length $y - 4$. As the length is four times the width, we know that $y - 4 = 4(x - 4)$, so $y = 4x - 12$.

We can then substitute this into the previous equation, to find that $x(4x - 12) = 1080$. Dividing by 4 and expanding the brackets gives $x^2 - 3x - 270 = 0$. This equation can then be factorised as $(x - 18)(x + 15) = 0$, so $x = 18$ or $x = -15$. As x is the width of the lowest layer, it cannot be negative, so $x = 18$.

Then $y = 4x - 12 = 4 \times 18 - 12 = 60$.

Suppose there are n layers in the pile. Then the top layer has dimensions $(x - n + 1) \times (y - n + 1)$. Since there are 400 marbles, this tells us that

$$(19 - n)(61 - n) = 400.$$

Expanding the brackets and simplifying,

$$n^2 - 80n + 759 = 0.$$

This factorises as

$$(n - 69)(n - 11) = 0.$$

Therefore $n = 69$ or $n = 11$. Since the dimensions of the top layer are positive, $19 - n > 0$, so $n < 19$. Hence $n = 11$.

This means the middle layer is the sixth layer, which has $(19 - 6)(61 - 6) = 13 \times 55 = 715$ marbles in it.

3. Xavier is twice as old as Yvette was when Zara was one year younger than Xavier is now.

Yvette is half as old as Zara will be when Xavier is ten years older than Yvette is now.

Zara is one and a half times as old as Xavier was when Yvette was ten years younger than Zara is now.

How old are Xavier, Yvette and Zara?

Answer Xavier is 18, Yvette is 16 and Zara is 24

Solution

Let us denote Xavier's current age by x, Yvette's current age by y and Zara's by z. We then need to consider each of the three pieces of information that we have been given:

(a) Zara was one year younger than Xavier is now $z - x + 1$ years ago (since we can go back z years to when Zara was born, then forwards $x - 1$ years to the correct age). At that point, Yvette's age was $y - (z - x + 1) = y + x - z - 1$. Since this is half of Xavier's current age, we know that $x = 2(y + x - z - 1)$. This simplifies to give

$$2z + 2 = x + 2y.$$

(b) Xavier will be ten years older than Yvette is now in $y + 10 - x$ years. At that point, Zara will be aged $z + y + 10 - x$, which is twice Yvette's current age. Therefore $2y = z + y + 10 - x$ or

$$x + y = z + 10.$$

(c) Yvette was ten years younger than Zara is now $y - z + 10$ years ago. At that point, Xavier was aged $x - (y - z + 10) = x - y + z - 10$. Since Zara's age is one and a half times this, we know that $z = \frac{3}{2}(x - y + z - 10)$, which rearranges to give

$$3y + 30 = 3x + z.$$

We now have three simultaneous equations that we can solve. From the second, we know that $z = x + y - 10$. Substituting into the other two gives:

$$2(x + y - 10) + 2 = x + 2y;$$
$$3y + 30 = 3x + (x + y - 10).$$

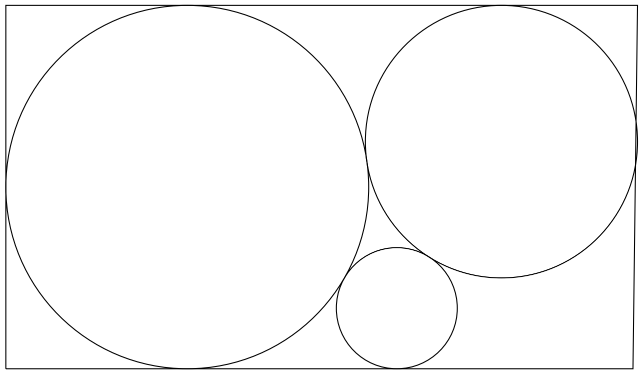
Simplifying gives:

$$x = 18;$$
$$2y + 40 = 4x.$$

Substituting the value of x into the second equation yields $2y + 40 = 72$, so $y = 16$. Then, using $z = x + y - 10$, we have $z = 18 + 16 - 10 = 24$.

Therefore Xavier is 18, Yvette is 16 and Zara is 24.

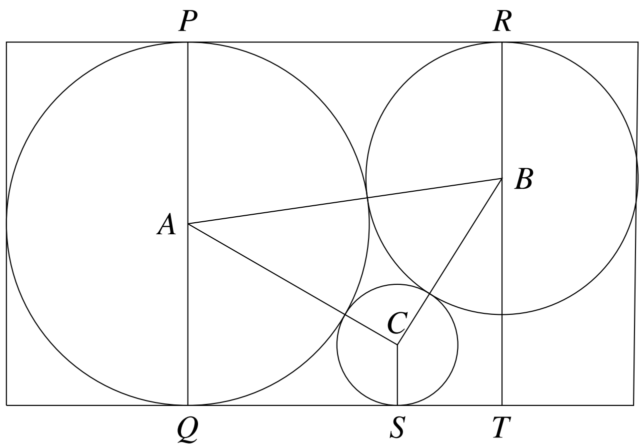
4. Three circles fit inside a rectangle as shown. Each circle touches the other two. The larger two circles touch two sides of the rectangle, and the smaller circle touches one side. If the radii of the larger circles are 12cm and 9cm, what is the radius of the smallest circle?



ANSWER 4cm

SOLUTION

Let A be the centre of the largest circle, B the next and C the smallest. Let P and Q be the points where the largest circle touches the top side and the bottom side, respectively, of the rectangle. Let the second smallest circle touch the top of the rectangle at R and let the smallest circle touch the bottom of the rectangle at S. Let T be the point where RB meets the bottom of the rectangle. The configuration is shown in the following diagram.



Since the sides of the rectangle are tangent at P, Q, R and S to the circles indicated, this means each of PQ, CS and RT are perpendicular to sides of the rectangle.

Let the radius of the small circle be r. Let the length QS be x, and the length ST be y.

Now let us calculate the length PQ. Using the trapezium $PABR$, which has right angles at P and R, we can get a right angled triangle with hypotenuse AB and other sides of length PR and $PA - BR$. Then Pythagoras' theorem tells us that $PR = \sqrt{AB^2 - (PA - BR)^2}$. Using the radii of the circles, $PA = 12$, $BR = 9$ and $AB = 9 + 12 = 21$. Hence

$$PR = \sqrt{21^2 - (12 - 9)^2} = \sqrt{441 - 9} = \sqrt{432} = 12\sqrt{3}.$$

But, as $PQTR$ is a rectangle, it follows that $x + y = 12\sqrt{3}$.

Now let us look at trapezium $AQSC$. We know that $QS = x$, $AQ = 12$, $CA = 12 + r$ and $Sc = r$. Then

Pythagoras' theorem tells us that

$$\begin{aligned} x = QS &= \sqrt{CA^2 - (AQ - CS)^2} = \sqrt{(12+r)^2 - (12-r)^2} = \sqrt{144 + 24r + r^2 - 144 + 24r - r^2} \\ &= \sqrt{48r} = 4\sqrt{3}\sqrt{r}. \end{aligned}$$

We can perform a similar calculation in trapezium $CSTB$. With $BC = 9 + r$, $TB = 24 - 9 = 15$, $ST = y$ and $CS = r$, we get

$$\begin{aligned} y = ST &= \sqrt{BC^2 - (TB - CS)^2} = \sqrt{(9+r)^2 - (15-r)^2} = \sqrt{81 + 18r + r^2 - 225 + 30r - r^2} \\ &= \sqrt{48r - 144} = 4\sqrt{3}\sqrt{r-3}. \end{aligned}$$

We can combine these calculations to obtain

$$12\sqrt{3} = x + y = 4\sqrt{3}\sqrt{r} + 4\sqrt{3}\sqrt{r-3}$$

Dividing by $4\sqrt{3}$ yields

$$\sqrt{r-3} = 3 - \sqrt{r}$$

and squaring this gives

$$r - 3 = 9 - 6\sqrt{r} + r.$$

This is equivalent to $6\sqrt{r} = 12$, so $r = 4$. Hence the small circle has radius 4 cm.

5. Twenty five of King Arthur's knights are seated around their round table. Three of them are chosen at random to slay a troublesome dragon. What is the probability that at least two of them had been seated next to each other?

Answer $\frac{11}{46}$

Solution

Each collection of three knights is equally likely, so we can calculate the number of sets of three knights such that at least two are next to each other, and then divide this by the number of ways of choosing three knights.

There are two different ways in which at least two of the three knights can sit adjacent to each other: either all three can sit in a row, or two can sit together and the third separately.

- If all three sit in a row, there are 25 choices for the middle knight in the row and the others are then fixed, so there are 25 ways of doing this.
- If two sit in a row and the third person is separate, then there are 25 adjacent pairs that can be chosen. The third person cannot be in this pair, nor can they be either of the two seats on the sides. This removes four of the 25 options, so there are 21 options remaining for the third person. This gives a total of 25×21 options.

This gives a total of $25 + 25 \times 21 = 25 \times 22$ ways of choosing the three knights such that two are adjacent.

Now, we need to consider the number of ways of choosing any three knights from around the table. There are 25 ways of choosing the first knight, then 24 for the second and 23 for the third. However, this means we choose each set of knights once in each of the six possible orders, so we need to divide by six to get the number of sets of three knights. Therefore there are

$$\frac{25 \times 24 \times 23}{6} = 25 \times 23 \times 4$$

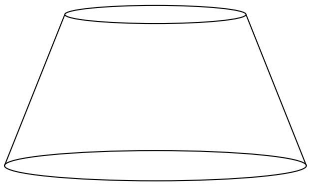
possible ways of choosing three knights.

Hence, the probability of choosing three knights at least two of which were seated next to each other is

$$\frac{25 \times 22}{25 \times 23 \times 4} = \frac{22}{4 \times 23} = \frac{11}{46}.$$

6. A frustrum is formed by taking a (right circular) cone and slicing the top off it. This frustrum has base radius 70mm, top radius 49mm and height 72mm.

If it is placed on a plane with its curved surface down and rolled without slipping, what is the outer radius of the circle it describes?



Answer 250 mm

Solution

> It is not necessarily clear out how to start this question. The key observation is that since the frustum is rolled without slipping, the radius of the circle described by any point on the frustum must be proportional to the radius of the frustum at that point.

Let R denote the radius of the circle described by the outer rim of the frustum.

Using Pythagoras' theorem, we can calculate that the sloping length of the cone is $\sqrt{(70-49)^2+72^2} = \sqrt{441+5184} = \sqrt{5625} = 75$. Hence the radius of the inner rim of the frustum is $R - 75$.

As the radii of the described circles are proportional to the radii of the rims, we know that

$$\frac{R}{70} = \frac{R-75}{49}.$$

Multiplying up, we have that

$$7R = 10R - 750.$$

Hence $3R = 750$, and $R = 250$, so the radius is 250 mm.

Alternative

> However, it is possible to make this solution simpler. Rolling the frustum is the same as rolling the cone formed by extending the curved surface up to the vertex.
>
> Since each point moves proportionately to the radius of the cone at that point, the vertex cannot move at all (as the radius there is zero). Thus the centre of the circle is at the vertex.
>
> This means that the radius of the outer rim is exactly the sloping height of the cone.

The cone without the frustum is similar to the full cone, so if the full cone has height h, we know that

$$\frac{h}{70} = \frac{h-72}{49}$$

Multiplying up gives

$$7h = 10h - 720$$

This means $h = 240$. Since the base radius is 70, by Pythagoras' theorem the sloping side has length $\sqrt{70^2 + 240^2} = \sqrt{4900 + 57600} = \sqrt{62500} = 250$ mm. Therefore this is the radius of the circle described by the outer rim.

7. For what integer values of N is $\frac{N^2-71}{7N+55}$ a positive integer?

ANSWER $N = -8$ or $N = 57$

SOLUTION

This follows on from Question 3 on the previous sheet: it might be worth reminding yourself of that question.

As $7N + 55$ is a factor of $N^2 - 71$, it must also be a factor of $7(N^2 - 71) = 7N^2 - 497$. Clearly $N(7N + 55) = 7N^2 + 55N$ is also a multiple of $7N + 55$.

This means the difference is also a multiple of $7N + 55$, so $55N + 497$ is a multiple of $7N + 55$.

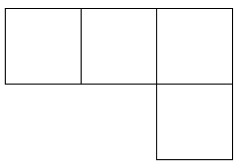
Then, $7(55N + 497)$ must be a multiple of $7N + 55$ and the same is true of $55(7N + 55)$, so the difference of these must also be a multiple. This difference is $(385N + 3479) - (385N + 3025) = 454$.

The factors of 454 are $\pm1, \pm2, \pm227, \pm454$. This means that $7N$ is one of $-509, -282, -57, -56, -54, -53, 172, 399$. Of these, only -56 and 399 are multiples of 7, so N must be -8 or 57.

When $N = -8$, $\frac{N^2-71}{7N+55} = \frac{64-71}{-56+55} = \frac{-7}{-1} = 7$, so this is a valid solution. When $N = 57$, $\frac{N^2-71}{7N+55} = \frac{3249-71}{399+55} = \frac{3178}{454} = 7$, so this is also a valid solution.

Therefore $N = -8$ or $N = 57$.

8. A tetromino is a two dimensional shape consisting of four squares joined together along their common edges. Two tetrominos are considered the same if one can be rotated onto the other.



(a) Prove that there are seven different tetrominoes.

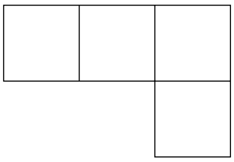
(b) Is it possible to fit the seven tetrominoes into a 7×4 portion of a chessboard?

ANSWER

(b) No

SOLUTION

(a) To show that there are at least seven tetrominoes, we can simply draw them.



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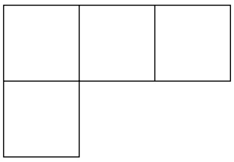

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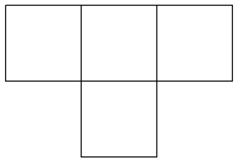

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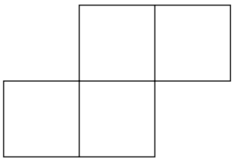

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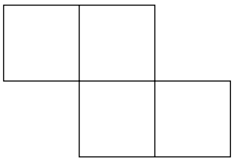

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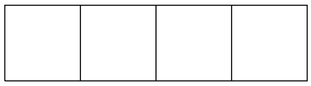

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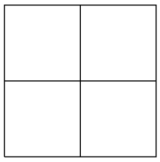
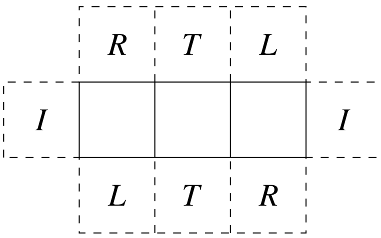
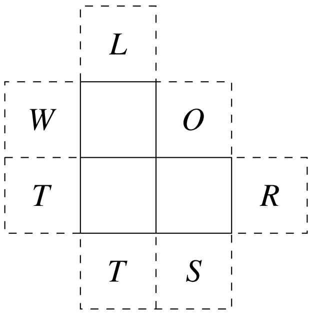


O

Then we need to check that we cannot make any more. We can do this by considering the triominoes (made of three squares) and asking where we could add an extra square. There are only two triominoes: if two squares are adjacent the third square must make either a line of three or a corner.

The diagram below shows these two triominoes and all the possible positions for the fourth square. In each of these is the letter corresponding to the tetromino that is formed. We see that no new tetrominoes arise.

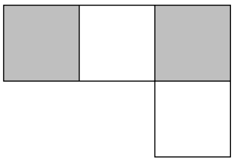




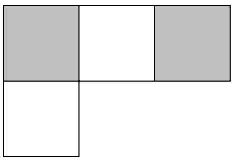
(b)

> If you try this a few times, you will find that you cannot get all seven of the shapes to fit into the rectangle. It turns out that you can get close but that you will always have one square in the wrong place.
>
> It might help if you look at the colour of the squares that are extra and missing. Every time the square missing has the opposite colour to the extra square. This means it seems like we should count the squares that are white and those that are black.

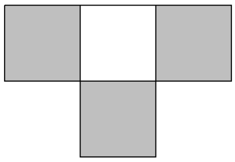
Consider colouring the board that the tetrominoes are placed on with the standard chessboard colouring. For each of the tetrominoes, we can use this to colour in their squares. We don't know which colour any particular square will be on, but we do know which squares will be on opposite colours



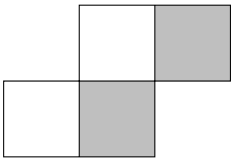
R



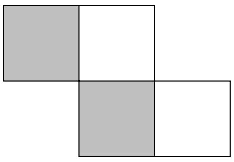
L



T

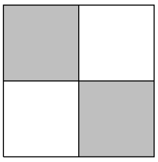


S



W

I



O

In each of these tetrominoes, there are the same number of black and white squares covered, except for T. This means that, however they are placed on the grid, they must cover an equal number of each colour of square.

The shape T must either cover three white squares and one black square or three black squares and one white square. This means a different number of white and black squares must be covered by all the tetrominoes altogether.

However, a 7×4 rectangle covers an equal number of white and black squares, so the whole rectangle cannot be covered.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Julia Robinson

Sheet 8

Solutions and comments

This programme of the Mentoring Scheme is named after Julia Robinson (1919–1985).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Robinson_Julia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

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University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 0113 343 5500 mentoring@ukmt.org.uk www.ukmt.org.uk

> **1.** Sophie sat several spelling tests during the last school year. In the penultimate test she scored 98 and her average (mean) score for the year so far increased by 1. In the last test she scored 70 and her average score decreased by 2.
>
> How many tests did she have?

Answer 10

Solution

Say that Sophie had n tests during the year and that her average score before the penultimate test (and so after $n - 2$ tests) was s. Then after $n - 2$ tests she had scored $(n - 2)s$ points in total.

After the penultimate test, her total score was $(n - 2)s + 98$. But we also know that her average score increased by 1, so her total score can also be written as $(n - 1)(s + 1)$. So $(n - 2)s + 98 = (n - 1)(s + 1)$. Rearranging this, we get

$$s + n = 99. \tag{1}$$

After the last test, Sophie's total score was $(n - 2)s + 98 + 70 = (n - 2)s + 168$. This time, her average score decreased from $s + 1$ (her average after the penultimate test) by 2, so we can also express her total score as $n(s - 1)$. So $(n - 2)s + 168 = n(s - 1)$, from which we obtain

$$2s - n = 168. \tag{2}$$

So we have two simultaneous equations (1) and (2) in the two unknowns n and s. Subtracting equation (2) from twice equation (1), we get $3n = 30$, so $n = 10$. That is, Sophie had 10 tests.

Remark

We may also then use equation (1) to find that her average score before the penultimate tests was $s = 99 - 10 = 89$.

> **2.** Find all triples of real numbers such that multiplying any two in a triple and adding the third always gives 1.

Answer $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $\left(\frac{1}{2}(\sqrt{5} - 1), \frac{1}{2}(\sqrt{5} - 1), \frac{1}{2}(\sqrt{5} - 1)\right)$ and $\left(\frac{1}{2}(-\sqrt{5} - 1), \frac{1}{2}(-\sqrt{5} - 1), \frac{1}{2}(-\sqrt{5} - 1)\right)$.

Solution

Suppose the triple of numbers is (x, y, z). Then the question tells us that they satisfy the following equations.

$$\begin{aligned} xy + z &= 1 \\ yz + x &= 1 \\ zx + y &= 1 \end{aligned}$$

Equating the left-hand sides of the first two equations shows that $xy + z = yz + x$. This rearranges as $xy - zy + z - x$, which in turn factorises as $(x - z)(y - 1) = 0$. This tells us that either $x = z$ or $y = 1$.

In the first case, we get $xy + x = 1$ and $x^2 + y = 1$. The difference between these equations yields $x^2 - xy + y - x = 0$, which factorises as $(x - y)(x - 1) = 0$. If $x = y$, then the first of the original equations

becomes $x^2 + x = 1$. We can solve this to obtain

$$x = y = z = \frac{\pm\sqrt{5} - 1}{2}.$$

If $x = 1$, then $y = 0$, and we obtain the triple $(1, 0, 1)$.

In the second case, we get from the third of the original equations that $xz = 0$, so either $x = 0$ or $z = 0$. If $x = 0$, the first equation tells us that $z = 1$. If $z = 0$, the second equation tells us that $x = 1$. These two results furnish the triples $(0, 1, 1)$ and $(1, 1, 0)$.

Therefore, there are five triples of solutions: $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $\left(\frac{1}{2}(\sqrt{5} - 1), \frac{1}{2}(\sqrt{5} - 1), \frac{1}{2}(\sqrt{5} - 1)\right)$ and $\left(\frac{1}{2}(-\sqrt{5} - 1), \frac{1}{2}(-\sqrt{5} - 1), \frac{1}{2}(-\sqrt{5} - 1)\right)$.

3. I want to use 1×1 square tiles to cover a 3×5 rectangular panel on my kitchen wall. I shall use yellow and green tiles; I have plenty of each colour. I want each 2×2 square, formed by four tiles with a common vertex, to have two yellow tiles and two green tiles.

How many such tiling patterns are there?

Answer 38.

Solution

Let's try tiling some smaller areas first and then build up to the 3×5 rectangle.

For a 3×1 rectangle, the possible tilings are as follows.

Y	Y	Y	Y	G	G	G	G
Y	Y	G	G	Y	Y	G	G
Y	G	Y	G	Y	G	Y	G
A	B	C	D	E	F	G	H

> The tilings are labelled so that they can be referred to later.

So there are 8. It is easy to work this out without showing all the possibilities: the restriction about 2×2 squares makes no difference here, so any tiling is possible. Since there are 2 possibilities for the colour of each tile, there are $2 \times 2 \times 2 = 8$ possible tilings.

Let us think of this 3×1 rectangle as the first column of a 3×2 rectangle. If the tiling of the first column contains two adjacent tiles of the same colour, then we have no choice about the tiling of the second column.

> Look at one of the examples above if you do not see this immediately.

Also, the second column will again have two adjacent tiles of the same colour. There are 6 tilings of the first column where this happens (A, B, D, E, G, and H). In the other two cases (C and F), there are two possibilities for the second column: either C or F can appear as the second column in either case. So we can count the total number of tilings of a 3×2 rectangle: there are $6 \times 1 + 2 \times 2 = 10$ such.

We now see how can we can count the tilings of a 3×5 rectangle.

If the first column is A, B, D, E, G or H (with at least two adjacent tiles of the same colour), then there is only one possible tiling of the rest, so this gives 6 possible tilings.

If the first column is C or F, then each of the subsequent columns is also C or F, and all these combinations are possible. This gives $2^5 = 32$ possible tilings.

So there are, in total, 38 possible tilings.

Remark

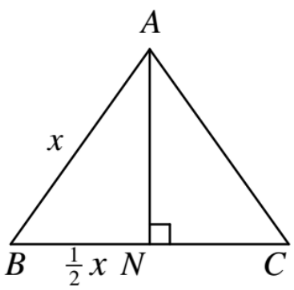
It is possible to argue in a similar way starting with the 1×5 tilings that form the rows.

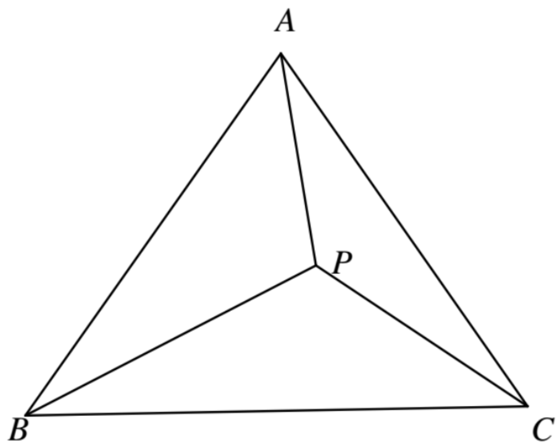
Another method is to consider just the middle row: there are $2^5 = 32$ possibilities, all but two of which determine everything else.

4. P is a point inside an equilateral triangle such that the distances from P to the sides of the triangle are 3, 4 and 5 respectively. Find the area of the triangle.

Solution

> We know that we can calculate the area of an equilateral triangle from its side length, and the area of any triangle from its height and base. This gives two ways to calculate the area of the equilateral triangle and equating these should give the side length and hence the area.





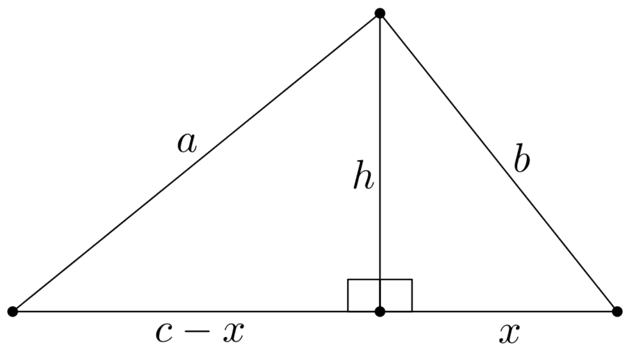
Suppose that the equilateral triangle ABC has side length x and let AN be the perpendicular from A to BC.

Then BN has length $\frac{x}{2}$ and hence, by applying Pythagoras' Theorem to the triangle ANB, it follows that $(\frac{1}{2}x)^2 + AN^2 = x^2$. This equation yields $AN = \frac{\sqrt{3}}{2}x$ and hence the area of triangle ABC is $\frac{1}{2} \times x \times \frac{\sqrt{3}}{2}x = \frac{\sqrt{3}}{4}x^2$.

On the other hand, the triangle ABC is divided into three triangles PAB, PAC, PBC each with base x and heights 3, 4 and 5 respectively. Thus the total area of these three triangles is $\frac{1}{2}\times 3\times x+\frac{1}{2}\times 4\times x+\frac{1}{2}\times 5\times x = 6x$. Since this is the area of triangle ABC, we have that $\frac{\sqrt{3}}{4}x^2 = 6x$. Therefore, as x cannot be 0, $x = \frac{24}{\sqrt{3}} = 8\sqrt{3}$. So the area of triangle ABC is $6 \times 8\sqrt{3} = 48\sqrt{3}$.

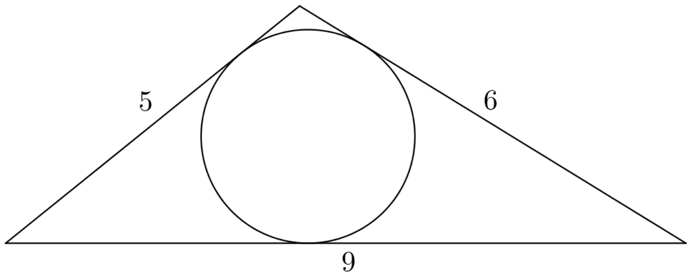
5. (a) A triangle has side lengths a, b and c. Denote by s the *semiperimeter* of the triangle (half the length of the perimeter). Use the following diagram to prove that the triangle has area

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$



This result is called *Heron's Formula*.

(b) A triangle with side lengths 5, 6 and 9 is inscribed in a circle. What is the radius of the circle?



Solution

(a) The area of the triangle is $A = \frac{1}{2}ch$. Since the semiperimeter is half the perimeter, we know that $s = \frac{1}{2}(a+b+c)$. We want to find the area of the triangle in terms of a, b and c only, so we need to eliminate h from the first equation.

Using Pythagoras' Theorem on the right-hand right-angled triangle gives $x = \sqrt{b^2 - h^2}$. Applying Pythagoras' Theorem to the left-hand right-angled triangle gives $(c-x)^2 + h^2 = a^2$. Combining these, we have

$$\left(c - \sqrt{b^2 - h^2}\right)^2 + h^2 = a^2.$$

Expanding the squares yields

$$c^2 - 2c\sqrt{b^2 - h^2} + b^2 - h^2 + h^2 = a^2.$$

Then the h^2 terms cancel and rearrangement produces

$$\sqrt{b^2 - h^2} = \frac{c^2 + b^2 - a^2}{2c}.$$

Again, squaring and rearranging gives

$$\begin{aligned} h^2 &= b^2 - \left(\frac{c^2+b^2-a^2}{2c}\right)^2 \\ &= \frac{4b^2c^2 - \left(c^4+b^4+a^4-2a^2b^2-2a^2c^2+2b^2c^2\right)}{4c^2} \\ &= \frac{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}{4c^2}. \end{aligned}$$

This can be substituted into the area formula to give

$$\begin{aligned} A &= \frac{1}{2}ch \\ &= \frac{1}{2}c\sqrt{\frac{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}{4c^2}} \\ &= \sqrt{\frac{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}{16}}. \end{aligned}$$

> You could at this point factorise the expression here, but I think it is easier to multiply out the expression given and to check that it is equal to the one that we have just found.

Now

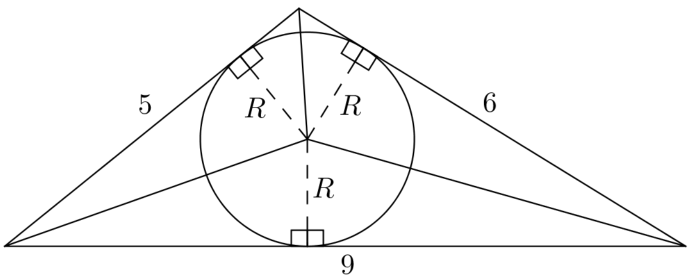
$$\begin{aligned} \sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)} \\ &= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)} \\ &= \sqrt{\frac{1}{16}(a+b+c)(b+c-a)(a-b+c)(a+b-c)}. \end{aligned}$$

The difference of two squares factorisation is a useful guide to multiplying out these brackets. We have

$$\begin{aligned} \sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{\frac{1}{16}(b+c+a)(b+c-a)(a+(c-b))(a-(c-b))} \\ &= \sqrt{\frac{1}{16}\left((b+c)^2-a^2\right)\left(a^2-(c-b)^2\right)} \\ &= \sqrt{\frac{1}{16}\left(2bc+(b^2+c^2-a^2)\right)\left(2bc-(b^2+c^2-a^2)\right)} \\ &= \sqrt{\frac{1}{16}\left(4b^2c^2-(b^2+c^2-a^2)^2\right)} \\ &= \sqrt{\frac{1}{16}\left(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4\right)}. \end{aligned}$$

This is exactly the same result as we obtained above, so Heron's formula is proved.

(b) Since all three sides of the triangle are tangent to the circle, the radii from the centre of the circle to these points are perpendicular to the sides of the triangle. This leads to the following diagram.



The original triangle has been split into three separate triangles, with bases 5, 6 and 9 but all having height R. This means the total area is

$$A = \frac{1}{2} \times 5R + \frac{1}{2} \times 6R + \frac{1}{2} \times 9R = 10R.$$

The semiperimeter is $\frac{1}{2}(5 + 6 + 9) = 10$ so, using Heron's formula, the area is

$$A = \sqrt{10(10-5)(10-6)(10-9)} = \sqrt{10 \times 5 \times 4 \times 1} = \sqrt{200} = 10\sqrt{2}.$$

Therefore $10R = 10\sqrt{2}$ and we conclude that $R = \sqrt{2}$.

6. Given a finite string of 0s and 1s, I have three possible operations that I may apply:

i. the substring 01 may be replaced by 100, and *vice versa*;

ii. the substring 10 may be replaced by 111, *vice versa*; and

iii. the substring 11 may be replaced by 000, and *vice versa*.

(a) Using these operations, is it possible to transform 11001 into 0001000?

(b) Is it possible to transform 0110 into 0110010?

Answer (a) Yes. (b) No.

Solution

(a) Here is an example of a sequence of operations that take 11001 into 0001000. (The substrings to be replaced are underlined.)

$$1\underline{100}1 \to 10\underline{11} \to \underline{10}000 \to \underline{11}1000 \to 0001000.$$

(b) Notice that the operations (i), (ii) and (iii) do not change the parity of the number of 1s in the string. That is, if the string has an odd number of 1s then so does the string after the operation and if the string has an even number of 1s, then so does the resulting string.

But 0110 has an even number of 1s and 0110010 has an odd number of 1s. So it cannot be possible to transform one into the other.

> This is a good example of the use of an invariant.
>
> We noticed that the parity of the number of 1s in the string is invariant under the operations (i), (ii) and (iii). Therefore, if two strings have different values of this invariant, then one cannot be obtained from the other.
>
> This idea is essentially quite simple, but is very useful.

|| ||

7. (a) i. Prove that a number is divisible by 2 exactly when the final digit is even. Prove that a number is divisible by 5 exactly when the final digit is divisible by 5.

ii. Prove that a number is divisible by 4 exactly when the two digit number formed by the last two digits is divisible by 4. Prove that a number is divisible by 8 exactly when the three digit number formed by the last three digits is divisible by 8.

iii. Prove that a number is divisible by 3 exactly when the sum of the digits is divisible by 3. Prove that a number is divisible by 9 exactly when the sum of the digits is divisible by 9.

(b) Can you find a number "$abcdefghij$" using each of the digits 0 to 9 once each such that the number formed by the first n digits is divisible by n for each $1 \leq n \leq 10$? (So, "ab" must be divisible by 2, for example, and "$abcdefg$" must be divisible by 7.)

Solution

(a) Let us write the number as "$a_n a_{n-1} \dots a_3 a_2 a_1 a_0$". Then we can consider each condition in turn.

i. "$a_n \dots a_1 a_0$" = "$a_n \dots a_1 0$" + a_0 = $10 \times$ "$a_n \dots a_1$" + a_0.
Then, since both 2 and 5 are factors of 10, they are factors of "$a_n \dots a_1 a_0$" exactly when they are factors of a_0.

ii. "$a_n \dots a_2 a_1 a_0$" = "$a_n \dots a_2 00$" + "$a_1 a_0$" = $100 \times$ "$a_n \dots a_2$" + "$a_1 a_0$".
Then since 4 divides 100, 4 divides "$a_n \dots a_1 a_0$" exactly if it divides "$a_1 a_0$".

"$a_n \dots a_3 a_2 a_1 a_0$" = "$a_n \dots a_3 000$" + "$a_2 a_1 a_0$" = $1000 \times$ "$a_n \dots a_3$" + "$a_2 a_1 a_0$".
Then since 8 divides 1000, 8 divides "$a_n \dots a_1 a_0$" exactly if it divides "$a_2 a_1 a_0$".

iii. For these conditions, it is worth noticing that $10^r = 99 \dots 99 + 1$, and so has remainder 1 when divided by either 3 or 9. This is the key fact we will use for answering this question.

"$a_n a_{n-1} \dots a_1 a_0$" = $10^n a_n + 10^{n-1} a_{n-1} + \dots + 10 a_1 + a_0$. Then $10^r - 1 = 99 \dots 9$, which is divisible by 9, so we can rewrite "$a_n a_{n-1} \dots a_1 a_0$" as $(a_n + a_{n-1} + \dots + a_0) + (10^n - 1)a_n + \dots + (10 - 1)a_1$. All of the terms are certainly divisible by 9 (and therefore by 3) except for the first. This tells us that the remainder is the same as that of the first term, which implies what we wanted to prove.

> There are many different ways of approaching part b of this question - in particular you can use the divisibility facts in many different ways in order to reduce the number of possibilities dramatically.

(b) Suppose we write the number as "$abcdefghij$". Since the full number is divisible by 10, the last digit must be 0, so $j = 0$.

Then, "$abcde$" is divisible by 5, so e must be either 5 or 0. But 0 has been taken already, so $e = 5$.

Then, since each of "ab", "$abcd$", "$abcd5f$" and "$abcd5fgh$" are even, b, d, f and h are all even, so are between them are 2, 4, 6 and 8. This means the other digits are all odd.

The table below shows the options at this stage.

a	b	c	d	e	f	g	h	i	j
1	2	1	2	5	2	1	2	1	0
3	4	3	4		4	3	4	3	
7	6	7	6		6	7	6	7	
9	8	9	8		8	9	8	9	

Since "$abcd$" and "$abcdefgh$" are both divisible by 4, this means "cd" and "gh" must both be divisible by 4. Since c and g are both odd, d and h can both only be 2 or 6. Then b and f must be either 4 and 8. This gives the table following.

a	b	c	d	e	f	g	h	i	j
1	4	1	2	5	4	1	2	1	0
3	8	3	6		8	3	6	3	
7		7				7		7	
9		9				9		9	

Then, since "abc" and "$abcdef$" are divisible by 3, this means $a+b+c$ and $a+b+c+d+e+f$ are divisible by 3. Hence $d+e+f$ is divisible by 3. This means the sum must be $2+5+8$ or $6+5+4$. Then the solution is one of the two possibilities below.

$$*4*258*6*0$$
$$*8*654*2*0$$

Here each of the $*$'s represents one of the digits 1, 3, 7 and 9.

However, we also know that "$abcdefgh$" is divisible by 8, so "fgh" is divisible by 8. If $h=2$, then $f=4$, so $g=3$. If $h=6$, then $f=8$, so $g=1$ or $g=9$. This yields the following four options.

$$*8*65432*0$$
$$*8*65472*0$$
$$*4*25816*0$$
$$*4*25896*0$$

Since "$abcdefghi$" is divisible by 9 and therefore by 3, this implies $a+b+c+d+e+f+g+h+i$ is divisible by 3. Consequently $g+h+i$ is also divisible by 3. This gives the following cases to examine.

- $*8*65432*0$: $i=1$ or $i=7$
- $*8*65472*0$: $i=3$ or $i=9$
- $*4*25816*0$: no possible value for i.
- $*4*25896*0$: $i=3$

Now we are left with the ten possibilities listed below.

7896543210
9876543210
1896543270
9816543270
1896547230
9816547230
1836547290
3816547290
1472589630
7412589630

Checking divisibility by 7 yields just one possible solution: 3816547290. This can be rechecked to ensure it does meet all the required conditions and consequently is the unique solution.

8. (a) Let p be a prime. Suppose that a, $a + d$, $a + 2d$, ..., $a + (p - 1)d$ is a sequence of equally spaced integers (with $d > 0$). Show that either d is divisible by p or that all the numbers in the sequence have different remainders when divided by p.

(b) A sequence a, $a + d$, $a + 2d$, ..., $a + (n - 1)d$ with $d > 0$ is called an *arithmetic progression*. What is the smallest possible value of the last term of such a sequence of length 5 where all the members are prime?

Hint: Can you prove that d must be divisible by both 2 *and* 3*? When is d not divisible by* 5*?*

Solution

(a) Suppose that any two of the elements of the sequence, $a + rd$ and $a + sd$ say, have the same remainder when divided by p. This means p must divide $(a + rd) - (a + sd) = d(r - s)$ Since p is prime, p must divide one of d or $r - s$. Since r and s are less than p apart and are different, p does not divide $r - s$, so p must divide d.

(b)

> The idea here is to make use of part i. Suppose that q is one of the primes 2, 3, 5. The idea is to restrict the possibilities for the sequence. The first is that one of the first q elements must be divisible by q: note that they all have different remainders. The alternative is that d must be divisible by q. This means we can restrict the values of d to be divisible by some of these primes.

Suppose the five primes are a, $a + d$, $a + 2d$, $a + 3d$ and $a + 4d$.

What happens if d is not even? Part i tells us that a and $a + d$ have different remainders when divided by 2, so one of them is even. The same is true of $a + 2d$ and $a + 3d$. Only one of these even numbers can be 2, so the other would not be prime, which would be a problem. This contradiction means 2 must divide d.

Suppose d is not divisible by 3. Notice that $a + 2d \geq 2d \geq 4$. We also know that $a + 2d$, $a + 3d$ and $a + 4d$ all have different remainders when divided by 3, and all are larger than 3. This means they could not all be prime, which would be a problem. Therefore d is divisible by 3. Since we already know that it is divisible by 2, we conclude that d is divisible by 6.

This means that the smallest possible prime number for the first term is 5, and the smallest possible value of d is 6. This leads to the sequence 5, 11, 17, 23 and 29, all of which are prime. Therefore the smallest possible value for the largest term is 29.